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INTRODUCTION

Any deformation of a solid R in 3-space which is not a rigid motion is accompanied by "strains", that is by changes in line-elements. Let two particles of the solid have the positions X,Y, a distance $\overline{XY}$ apart, in its original position, and let x,y be the positions of the same particles in the deformed state, a distance $\overline{xy}$ apart. The <u>non-rigidity</u> of the deformation is then measured by the amount the quotient

$$\overline{xy}/\overline{XY} = Q(X,Y)$$

differs from 1 for any X,Y in R. The state of strain of the solid due to the deformation is measured by the deviation from 1 of the same quotient, only formed for "neighboring" points X,Y. More precisely we define

$$M = \sup_{X \in R} \ \overline{\lim_{Y \to X}} \ Q(X,Y) \ , \qquad m = \inf_{X \in R} \ \overline{\lim_{Y \to X}} \ Q(X,Y) \ ;$$

then the amounts by which M and m differ from 1 give us a measure for the maximum strain accompanying the deformation.

The present paper is concerned with the range of values Q(X,Y) can assume for arbitrary deformations of a solid occupying a region R in the undeformed state, if the values m,M which limit the strains, are prescribed. We restrict ourselves here to deformations that locally are homeomorphisms. We call mappings of R for which the quantities M and m have finite

positive values "quasi-isometric", and more precisely (m,M)-isometric.[1] In one-dimensional space the quasi-conformal mappings of an interval are the Lipschitz-continuous mappings with a Lipschitz-continuous inverse. Here we always have $m \leqq Q(X,Y) \leqq M$, which is essentially the statement of the mean value theorem of calculus. In higher dimensions Q does not have to lie at all between m and M, and correspondingly large deformations can be compatible with small strains. What limitations there are on the values Q depends completely on the shape of R. If R is convex we are at least sure that $Q(X,Y) \leqq M$ (Cf. Lemma I), essentially by virtue of the triangle inequality. But it is not true, even for convex R, that $Q(X,Y) \geqq m$ has to be satisfied. Still, some positive lower bounds for Q can be found. What they are depends on how "bulky" the solid and how large the strains. The ends of a thin rod can be brought together by deformations involving only small strains, but a bulky solid has "stiffness" in the sense that any relative change in distance of two points

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1. The term "quasi-isometric" (used incidentally in a different sense by other authors) is chosen in analogy to "quasi-conformal", which, subject to appropriate regularity conditions, would be defined by the requirement that

$$\sup_{X \in R} \frac{\overline{\lim}_{Y\to X} \; Q(X,Y)}{\underline{\lim}_{Y\to X} \; Q(X,Y)}$$

has a finite value. A quasi-isometric mapping also is quasi-conformal.

is accompanied necessarily by strains of roughly the same order of magnitude.[1] This at least is the case for sufficiently small strains. If strains are increased slowly a (not sharply defined) point will be reached where the relative changes in distance can become very large compared to the maximum strains. For convex bodies this occurs when the strains are of the size of the square of the thickness length ratio. (Cf. theorem X). It is plausible that the order of magnitude of strains at which the solid loses its stiffness is the same as that at which buckling can occur. At least the possibility of obtaining large deformations for relatively small strains and, hence, relatively small strain energy, should enhance the possibility of having a variety of equilibrium states. The precise strains or stresses needed to produce buckling depend, of course, on material constants and the precise ways loads are applied; but purely kinematic considerations of the type pursued here might give correct orders of magnitude. For Euler's Elastica, for example, one easily convinces oneself by dimensional arguments that indeed the strains accompanying buckling are of the order of the square of the thickness length ratio.

For the results discussed in the present paper the number of dimensions, as soon as it exceeds 1, is unessential. For that reason everything is proved for quasi-isometric mappings in Hilbert-space, and in the beginning more generally in Banach

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1. We are concerned here only with stiffness due to sheer bulk of a solid. The much more subtle phenomenon of stiffness in thin closed shells, is not considered.

space. A few of the theorems (notably theorem IV) had been given by the author in earlier papers (Cf. [4], [5]) for the case of euclidean space. Modified proofs appear here, not making use of compactness or of linear approximations. Though the results apply to mappings of regions in Hilbert spaces the actual arguments used lean heavily on plane geometry of circles, ellipses and convex sets in general, as the reader will gather from the accompanying figures.

The results given here can be looked at as multi-dimensional versions of the mean-value theorem of differential calculus. The quantities M and m are upper and lower bounds of Q(X,Y) for Y differing only infinitesimally from X. The first step taken (Cf. theorems I,II) is to show that they actually also are upper and lower bounds of Q(X,Y) for X and Y a finite distance apart, provided X and Y are sufficiently far removed from the boundary of the domain of the mapping. It is sufficient that X and Y belong to a ball of radius ρ which is such that the concentric ball of radius $\frac{M}{m}\rho$ lies in R. This implies that in the case that the domain of the quasi-isometric mapping is the whole space that Q(X,Y) lies between m and M for all X,Y. It also shows that in the case of an isometric mapping (m=M=1) the distance of any two points X,Y is preserved in the mapping, provided X and Y belong to one and the same ball contained in the domain R. Using a result of Mazur and Ulam one finds that a mapping f of a connected set R in Banach space that is locally a homeomorphism and for which

$$\lim_{Y \to X} \frac{|f(Y)-f(X)|}{|Y-X|} = 1 \quad \text{for all } X \text{ in } R$$

is affine and distance preserving (Cf. theorem III).

We have generally two types of statements about the quotient $Q(X,Y)$ when the upper and lower bounds M,m of $Q(X,Y)$ for $Y \to X$ are given. The first type of statement gives conditions on X and Y which assure that $m \leqq Q(X,Y) \leqq M$. The second kind of statement gives bounds for $Q(X,Y)$ valid for all X,Y in R. Theorem IV is of the first type. It assures us that $m \leqq Q(X,Y) \leqq M$ if the ellipsoid of revolution of foci X,Y and eccentricity m/M lies in the domain R. The main use made of this theorem and of its refinement theorem VI is to obtain upper and lower bounds for $Q(X,Y)$ for any X,Y in R, in case R is a ball in Hilbert-space. It turns out in particular: any (m,M)-isometric mapping of a ball in Hilbert-space is invertible when $M/m < \sqrt{2}$. Here, as elsewhere in this paper, no "best" results are obtained. In many cases estimates derived here give the correct order of magnitude of quantities but with constant factors that are unrealistically poor. There must be a largest universal constant γ such that (m,M)-isometric mappings of balls with $M/m < \gamma$ are invertible. It is proved here that this largest constant γ is not smaller than $\sqrt{2}$. Counterexamples show that it cannot exceed the value 2. It would be of interest to find the best constant, even for mappings of a disk in the plane, or even for conformal mappings. For conformal mappings the question would be to find the largest γ with the property that every conformal mapping f of a disk

for which

$$\frac{\text{Max } |f'|}{\text{Min } |f'|} < \gamma$$

is invertible. More generally any (m,M)-isometric mapping of a convex set R is invertible, if only M/m is sufficiently close to 1, (the required degree of closeness depending on the shape of R; (Cf. Corollary IX).

Of special interest are the regions R (called here "spheroids") that can be mapped quasi-isometrically and bi-uniquely onto balls. (For example, regions in euclidean space that can be mapped bi-uniquely on a ball by a mapping that has continuous first derivatives and a Jacobian bounded away from zero.) It is shown here that all open convex sets, and more generally all sets that are starshaped from all points of some ball, are spheroids. (Cf. Theorem VIII).

The last theorems taken up deal with bounds for $Q(X,Y)$ in the case of an (m,M)-isometric mapping of a convex set R in Hilbert-space. The problem is to get quantitative information on the "stiffness" of such sets (which also could be called "lack of flexibility"). First a measure for the stiffness of R with respect to two chosen points X,Y of R is defined. One takes the extreme values M' and m' of $Q(X,Y)$ for all possible (m,M)-isometric mappings of R, puts $M/m = (1+\varepsilon)^2$, $M'/m' = (1+\varepsilon')^2$, and defines the stiffness of R with respect to the points X,Y for given ε by $s(\varepsilon,R,X,Y) = \varepsilon/\varepsilon'$. The stiffness of R for maximum strain ε is then $s(\varepsilon,R) = \inf s(\varepsilon,R,X,Y)$ for X,Y ranging over R.

The stiffness depends on the shape of R. The only shape-factor that will be taken into account is the ratio α/β of the radii of an inscribed and concentric circumscribed sphere, which in euclidean space is also a measure for the "thickness-length ratio" of R. Stiffness ranges from the value 1 down to zero. It is likely to decrease with increasing maximum strain ε. For two given points X,Y of an open convex set R the stiffness $s(\varepsilon,R,X,Y)$ always has the value 1 when ε is sufficiently small. However, for a bounded open convex set R and any positive ε we can always find points X,Y for which $s(\varepsilon,R,X,Y) < 1$, that is $s(\varepsilon,R) < 1$ for $\varepsilon > 0$. (If R is the whole space then $s(\varepsilon,R) = 1$ for all ε.) Moreover for bounded open convex R the stiffness has the value zero as soon as the maximum strain ε exceeds the universal value $\sqrt{2} - 1$; that is for $M/m > 2$ we can construct (m,M)-isometric mappings that are not univalued in R. It is likely that for convex R the stiffness $s(\varepsilon,R)$ is close to 1 for all sufficiently small ε (for non-convex R the stiffness $s(\varepsilon,R)$ is less than 1, even for arbitrarily small ε). Only a weaker result is proved here, namely that $s(\varepsilon,R)$ has at least the value 1/2 for sufficiently small ε, more precisely for $\varepsilon << \beta^2/\alpha^2$. (Theorem X and Corollary X)

I. Quasi-isometric mappings in Banach space.

1. Regular mappings.

We consider a Banach space B with elements X,Y,... and another one, b, with elements x,y,... . Let x=f(X) be a mapping whose domain is an open set R in the space B and whose range lies in the space b .

The mapping f will be called *regular* if it is locally a homeomorphism. More precisely we define f to be regular if the following conditions are satisfied:

Ia) f is *continuous* in R .

Ib) f is *open*, i.e. maps open subsets of R into open sets.

Ic) f is *locally one-one*, i.e. for every $X \in R$ there is a neighbourhood of X contained in R with the property that distinct points in that neighbourhood are mapped by f into distinct points.[1]

A mapping X = G(x) whose domain is a set d in the space b and whose range lies in the set R is called an *inverse* of f, if

1) G is continuous in d

2) f(G(x)) = x for all x in d.

Regular mappings have local inverses, as is proved easily: *If $X^o \in R$ and $x^o = f(X^o)$ there exists an inverse G(x) of f defined in a neighbourhood of x^o which has the property that $G(x^o) = X^o$.* Inverses have continuation properties similar to analytic

1. In the special case where both spaces B and b are of the same finite dimension conditions Ia) and Ic) already imply Ib) by the *invariance of domain* theorem.

functions. In particular we can continue them along <u>rays</u> from a point x°. One verifies easily the following propositions: Let X° be a point of R and $x^{\circ} = f(X^{\circ})$. We consider a "ray" r from x°, i.e. a set of points x b of the form

$$x = x^{\circ} + \lambda y$$

where y is a fixed non-vanishing element of b, and λ runs through all non-negative real numbers. We denote for any positive number μ by r_{μ} the subset of r of points $x=x^{\circ}+\lambda y$ with parameter values λ satisfying $0 \leqq \lambda < \mu$. For $\mu = \infty$ we still define $r_{\mu} = r$. We consider inverses G of f defined on sets r_{μ} for which $G(x^{\circ}) = X^{\circ}$. There exist such inverses for μ sufficiently small. Moreover <u>all such inverses are restrictions to r_{μ} of one and the same global inverse</u> which we shall call $f^{-1}_{X^{\circ}}(x)$ and which has as its domain a set r_{ρ} where ρ is the largest possible μ (possibly infinite). <u>If</u> $Y = \lim_{\lambda \to \rho} f^{-1}_{X^{\circ}}(x^{\circ}+\lambda y)$ <u>exists at all it must be a boundary point of</u> R. The mapping $f^{-1}_{X^{\circ}}$ can be continued into a neighbourhood in b of any compact subset of r_{ρ}, i.e. for any μ with $0 < \mu < \rho$ there exists a positive δ and an inverse G(x) of f defined in the set

$$\{x: \ |x-x^{\circ}-\lambda y| < \delta \ \text{ for some } \lambda \text{ with } 0 \leqq \lambda \leqq \mu\}$$

such that $G(x) = f^{-1}_{X^{\circ}}(x)$ for $x \in r_{\mu}$.

We can then associate with every point $X^{\circ} \in R$ a unique inverse $f^{-1}_{X^{\circ}}(x)$ defined in a <u>star-shaped</u> region as follows: Let $x^{\circ} = f(X^{\circ})$. For every $y \in b$ with $|y|=1$ we consider the ray r

of points $x = x^{o} + \lambda y$ with $\lambda \geqq 0$. We take the global inverse $f^{-1}_{x^o}$ defined along the ray for $0 \leqq \lambda < \rho$ (where ρ depends on y). The union of all the sets r_ρ for varying unit vectors y forms the star s_{x^o} with the vertex x^o. This star is an open set. Each point x of s_{x^o} lies on a certain r_ρ in which there is defined a value of $f^{-1}_{x^o}(x)$. In this way we have defined uniquely a mapping $f^{-1}_{x^o}(x)$ which turns out to be continuous in s_{x^o}, and is an inverse of f with domain s_{x^o} with the property that $f^{-1}_{x^o}(x^o) = X^o$.

Let there be given an arc Γ in R, that is a set of points $X = H(\lambda)$ where H is a continuous function defined on an interval $0 \leqq \lambda \leqq \alpha$. Let $H(0) = X^o$. If then the image $f(\Gamma)$ lies completely in the star s_{x^o}, i.e. if $f(H(\lambda)) \in s_{x^o}$ for $0 \leqq \lambda \leqq \alpha$, then

$$f^{-1}_{x^o}(f(X)) = X$$

for all $X \in \Gamma$.

We also observe that because of the uniqueness of continuation of inverses along rays any two inverses of f defined in a convex set agree in all points of the set if they agree in a single point.

2. Definition of quasi-isometric mappings.

A mapping $x = f(X)$ defined in the open set R is called quasi-isometric (more precisely (m,M)-isometric) if it is regular and if there exist positive finite numbers m and M such that for

all X in R

IIa) $$\overline{\lim_{Y \to X}} \frac{|f(Y)-f(X)|}{|Y-X|} \leqq M$$

IIb) $$\underline{\lim_{Y \to X}} \frac{|f(Y)-f(X)|}{|Y-X|} \geqq m$$

The conditions Ia,b,c) and IIa,b) are not completely independent. Condition IIa) alone implies already the continuity of f in R.

It is clear that any inverse of an (m,M)-isometric mapping defined in an open set is again quasi-isometric, and more precisely is (M^{-1}, m^{-1})-isometric.

3. Examples of quasi-isometric mappings.

A particular type of quasi-isometric mappings that has received attention (see R. Nevanlinna [1]) are those of the form

$$f(X) = X - g(X) \tag{3.1}$$

where the domain of g is an open set R in the space B, the range of g is also in B, and where it is assumed that there exists a constant $q < 1$ such that for each X in R

$$\overline{\lim_{Y \to X}} \frac{|g(Y)-g(X)|}{|Y-X|} \leqq q \tag{3.2}$$

The mapping f can then be shown to be $(1-q, 1+q)$-isometric.

Other quasi-isometric mappings are those f(X) that at each point in R have a Frechet derivative f'(X), which depends continuously on X, has an inverse $(f'(X))^{-1}$ and is such that for

all $X \in R$

$$(3.)\qquad |f'(X)| \leqq M\ , \quad |f'(X))^{-1}| \leqq \frac{1}{m}\ .$$

(here the norms of f' and of its reciprocal are defined as usual for linear operators). In the special case where B and b are of the same finite dimension n and the mapping function has continuous derivatives the matrix $f'(X)$ is just the Jacobian matrix of the mapping. If here B and b are referred to the ordinary euclidean distance then necessary and sufficient for the mapping f to be (m,M)-isometric is that each eigenvalue λ of the symmetric matrix $f'^{T}f'$ satisfies $m^2 \leqq \lambda \leqq M^2$. The existence <u>almost everywhere</u> of a derivative f' with $m^2 \leqq \lambda \leqq M^2$ is necessary for an (m,M)-isometric mapping of euclidean space.[1]

Differentiable quasi-isometric mappings are always quasi-conformal in the sense that they take infinitesimal spheres into infinitesimal ellipsoids of bounded eccentricity. Differentiability is however not at all necessary for a quasi-isometric mapping. For example continuous piecewise linear mappings that do not differ too much from the identity are quasi-isometric.

Simple examples of quasi-isometric mappings are furnished by <u>conformal</u> mappings in the plane. If $F(Z) = F(X+iY)$ is an analytic function of the single complex variable $Z = X + iY$ then the mapping is (m,M)-isometric if

$$m \leqq |F'(Z)| \leqq M$$

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1. This follows from the theorem of Rademacher [9] about differentiability almost everywhere of Lipschitz continuous functions.

(where we assume, as always, that $0 < m \leqq M < \infty$).

4. Rigidity of quasi-isometric mappings.

A mapping $x = f(X)$ defined in a set in B will be called (m,M)-rigid, if for any two points X, Y of the set the inequalities

$$m|Y-X| \leqq |f(Y)-f(X)| \leqq M|Y-X| \tag{4.1}$$

are satisfied.

If B and b are one-dimensional euclidean spaces the (m,M)-isometric mappings f of an open interval are also (m,M)-rigid. For differentiable f this is immediate from the mean value theorem of differential calculus. For others it follows by applying lemma I below to f and to its inverse, which also has an interval as its domain.

In two dimensions already the notions of (m,M)-isometry and (m,M)-rigidity diverge from each other. Thus the conformal mapping $z = e^Z$ is $(1,2)$-isometric in the strip

$$0 < \operatorname{Re}(Z) < \log 2$$

but for $Z_1 = \frac{1}{2}\log 2 + \pi i$, $Z_2 = \frac{1}{2}\log 2 - \pi I$ we have

$$\frac{|z_1 - z_2|}{|Z_1 - Z_2|} = 0 < 1 .$$

Here we shall be interested in finding subsets of the domain R of an (m,M)-isometric mapping, in which the mapping is also (m,M)-rigid. In a sense this is a question of generalizing the mean value theorem of differential calculus to higher

dimensions. The quantity $|f(Y)-f(X)|/|Y-X|$ plays the role of the absolute value of the difference quotient. We try to bound it from above and below by upper and lower bounds for the "derivative" at points of the region.

The main tool here is a lemma that essentially shows the upper bounds of difference quotients and derivatives in convex sets to be identical:

Lemma I: *Let $x = f(X)$ be a mapping whose domain is a convex set R in a Banach space B and whose range lies in a Banach space b.*[1]

Let for each X in R

$$(4.2) \qquad \overline{\lim_{\substack{Y\to X\\ Y\in R}}} \frac{|f(Y)-f(X)|}{|Y-X|} \leqq M .$$

Then for any X,Y in R

$$|f(Y)-f(X)| \leqq M|Y-X| .$$[2]

Proof: Let X,Y be two points of R. Then the points

$$Z_\lambda = (1-\lambda)X+\lambda Y$$

belong to R for $0 \leqq \lambda \leqq 1$. We have in $\phi(\lambda) = f(Z_\lambda) = f((1-\lambda)X+\lambda Y)$ a function that maps the interval $0 \leqq \lambda \leqq 1$ into the space b and satisfies for arguments in that interval the

1. We do not assume here that the mapping is regular or that R is open.

2. A proof of this lemma is given by R. Nevanlinna [8]. A closely related statement is proved by A. K. Aziz and J. B. Diaz [1].

relation

$$(4.4)\qquad \overline{\lim_{\lambda\to\mu}} \frac{|\phi(\lambda)-\phi(\mu)|}{|\lambda-\mu|} \leqq M|Y-X| = \gamma$$

It follows that $\phi(\lambda)$ is continuous in the closed interval. We want to prove that

$$|\phi(1)-\phi(0)| \leqq \gamma .$$

If that were not the case there would exist a positive ε such that

$$(4.5)\qquad |\phi(1)-\phi(0)| > (1+\varepsilon)\gamma .$$

We consider the set $\sum$ of values λ with $0 < \lambda \leqq 1$ for which

$$|\phi(\lambda)-\phi(0)| > (1+\varepsilon)\gamma\lambda .$$

By (4.5) $\sum$ contains the point $\lambda = 1$. By (4.3) the points λ sufficiently close to 0 do not belong to $\sum$. Hence

$$\mu = \inf_{\lambda\in\sum} \lambda$$

satisfies $0 < \mu \leqq 1$. By continuity of the function $\phi(\lambda)$ the point μ does not belong to $\sum$, that is

$$|\phi(\mu)-\phi(0)| \leqq (1+\varepsilon)\gamma\mu .$$

But then we have by (4.3) for all λ sufficiently close to μ and greater than μ

$$\begin{aligned} |\phi(\lambda)-\phi(0)| &\leqq |\phi(\lambda)-\phi(\mu)| + |\phi(\mu)-\phi(0)| \\ &\leqq (1+\varepsilon)(\lambda-\mu)\gamma + (1+\varepsilon)\gamma\mu = (1+\varepsilon)\gamma\lambda . \end{aligned}$$

Hence μ cannot be the greatest lower bound of $\sum$, unless $\mu = 1$, in which case $\sum$ is empty.

5. Rigidity of mappings of balls.

Theorem I.

Let $x = f(X)$ be (m,M)-isometric in the open set R. Let X^{o} be a point of R and ρ its distance from the boundary of R (that is $\rho = \inf|Y-X^{o}|$ for $Y \in R$). Let $x^{o} = f(X^{o})$. Then the star $s_{X^{o}}$ contains the ball $|x-x^{o}| < m\rho$. In other words: If the ball of radius ρ and center X^{o} is contained in R then f has a univalued inverse f^{-1} mapping $f(X^{o})$ into X^{o} at least in the ball of radius $m\rho$ about $f(X^{o})$. Or again: If the regular mapping f magnifies infinitesimal balls at least m-fold in all directions then it also magnifies balls in R of finite radius at least m-fold in all directions.[1]

Proof:

Let along a ray $x = x^{o} + \lambda y$ (where $|y| = 1$, $\lambda \geqq 0$) from x^{o} the points with $0 \leqq \lambda < \mu$ be those belonging to the star $s_{X^{o}}$. These points form the subset r_{μ} of the ray. On r_{μ} we have defined the inverse $f^{-1}_{X^{o}}$ of f. Since $f^{-1}_{X^{o}}(x)$ is (M^{-1},m^{-1})-isometric in $s_{X^{o}}$ and r_{μ} is a convex subset of $s_{X^{o}}$ we have by lemma I, p.7

$$(5.1) \qquad |f^{-1}(x)-f^{-1}(y)| \leqq m^{-1}|x-y|$$

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1. For the special mappings of the form (3.1), (3.2) the theorem is proved (with a more precise estimate) by Nevanlinna [8]. Related theorems are given by E. H. Zarantonello [10], G. Minty [7], and F. E. Browder [3].

for any two points x, y of r_μ. It follows then from the completeness of the space B that in case μ is finite

$$\lim_{\lambda \to \mu -} f^{-1}_{X^0}(x^0+\lambda y) = Z \tag{5.2}$$

exists. Here Z can only be a boundary point of R, since otherwise $f^{-1}_{X^0}$ could be continued along the ray beyond the point $x^0 + \mu y$. Hence $|Z-X^0| \geqq \rho$. The inequality (5.1) yields for $y = x^0$ and $x = x^0 + \lambda y$ with $0 < \lambda < \mu$

$$|f^{-1}(x^0+\lambda y) - X^0| \leqq m^{-1}\lambda \ .$$

Hence also by (5.2)

$$|Z-X^0| \leqq m^{-1}\mu$$

and consequently $\rho \leqq m^{-1}\mu$. Thus along each ray from x^0 the inverse $f^{-1}_{X^0}$ can be continued at least a distance $m\rho$ which was to be proved.

Theorem II:

<u>If $f(X)$ is (m,M)-isometric in the ball</u> $|X-X^0| < \rho$, <u>then $f(X)$ is (m,M)-rigid (that is (3.1) holds) in the concentric ball</u> $|X-X^0| < \frac{m}{M}\rho$.

Proof:

Since the ball $|X-X^0| < \rho$ is convex we obtain immediately from Lemma I that

$$|f(Y)-f(X)| \leqq M|Y-X| \tag{5.3}$$

for any X, Y in the whole ball of radius ρ. In order to prove

the remaining inequality we observe that by theorem I we have in $f^{-1}_{X^o}(x)$ an (M^{-1},m^{-1})-isometric mapping defined in the ball $|x-x^o| < m\rho$, where $x^o = f(X^o)$. Applying (5.3) to that mapping we find that

$$(5.4) \qquad |f^{-1}(x)-f^{-1}(y)| \leqq m^{-1}|x-y|$$

for x,y in the ball of radius $m\rho$ about x^o. Moreover by theorem I applied to $f^{-1}_{X^o}$ this function maps the ball $|x-x^o| < m\rho$ onto a set in B containing the ball $|X-X^o| < M^{-1}m\rho$. Let then X,Y be any points in the ball $|X-X_o| < M^{-1}m\rho$. It is then possible to represent X and Y in the form $X = f^{-1}_{X^o}(x)$, $Y = f^{-1}_{X^o}(y)$ where $|x-x^o| < m\rho$, $|y-x^o| < m\rho$. By definition of inverse we have then $f(X) = x$, $f(Y) = y$. It follows then from (5.4) that

$$|X-Y| \leqq m^{-1}|f(X)-f(Y)| ,$$

which completes the proof.

<u>Corollary I.</u> <u>An (m,M)-isometric mapping of the whole space is a homeomorphism between the Banach spaces B and b, and is, moreover, (m,M)-rigid everywhere.</u>

Proof: If the domain R of f is the whole space B we have from theorem II for $\rho \to \infty$ that f is (m,M)-rigid everywhere. In particular $|f(Y)-f(X)| \neq 0$ for $|Y-X| \neq 0$, i.e. the mapping f is one-one. By theorem I the range of f contains arbitrarily large balls about one of its points, and hence is the whole space b.

Corollary II. If the mapping $f(X)$ has a continuous derivative $f'(X)$ in the ball $|X-X^o| < \rho$, and if for all X in the ball f' has an inverse satisfying $|(f')^{-1}| \geqq m > 0$, then f has an inverse $f^{-1}(x)$ for $|x-f(X^o)| < m\rho$.

5*. Isometric mappings.

We call a mapping $x = f(X)$ isometric, if it is (1,1)-isometric, i.e. if f is regular in an open set R and satisfies at each point X of R

(5*.1)
$$\lim_{Y \to X} \frac{|f(Y)-f(X)|}{|Y-X|} = 1 .$$

Similarly we call the mapping $f(X)$ rigid in a set, if it is (1,1)-rigid, or distance preserving, that is if

(5*.2)
$$|f(Y)-f(X)| = |Y-X|$$

for any X,Y of the set.

Given any ball $|X-X^o| < \rho$ contained in the domain of an isometric mapping f it follows from theorems I, II that f maps the ball one-one and rigidly onto the ball $|x-f(X^o)| < \rho$.

A theorem of Mazur and Ulam [6][1] asserts (in our terminology) that a rigid homeomorphism f between two Banach spaces B and b is affine ("linear" within a translation), that is we have

(5*.3)
$$f((1-\lambda)X+\lambda Y) = (1-\lambda)f(X)+\lambda f(Y)$$

for any points X,Y in B and any real λ.

1. See also Banach [2], pp. 166-8.

From this we easily prove:

Theorem III.

If $f(X)$ is isometric (in the sense used here) in an open connected set R then f coincides in R with an affine rigid mapping of the whole space.

Proof: The key point in the proof of Mazur and Ulam is the characterization[2] of the "midpoint" $T = \frac{1}{2}(X+Y)$ of two points X,Y in Banach space purely in terms of distances. One defines recursively the sets $\sum_n$ by

$$\sum\nolimits_1 = (Z:\ |Z-X| \leqq \tfrac{1}{2}|X-Y|)\ ,\quad |Z-Y| \leqq \tfrac{1}{2}|Y-X|)$$

$$\sum\nolimits_n = (Z:\ Z \in \sum\nolimits_{n-1}\ ,\ |Z-U| \leqq 2^{1-n}|Y-X| \text{ for all } U \in \sum\nolimits_{n-1}\ .$$

One verifies by induction that each set $\sum_n$ contains the point T and is symmetric with respect to T (that is, contains with any Z also $Z' = 2T-Z$). Since then also $|Z-T| \leqq 2^{-n}|Y-X|$ for all $Z \in \sum_n$ the midpoint T of X and Y is characterized uniquely as the point common to all sets $\sum_n$. We notice that all the sets $\sum_n$ for $n=1,2,\ldots$ lie in the ball with diameter XY, that is the ball $|Z-T| \leqq \frac{1}{2}|Y-X|$, and that in constructing $\sum_n$ we could restrict ourselves to points of that ball, and only make use of distances between points of that ball.

Let now T be a point of the domain R of our isometric mapping f, and let R contain a ρ-neighborhood of T. Let $t=f(T)$. Then f maps the ball $|Z-T| < \rho$ one-one rigidly onto the ball $|z-t| < \rho$. Let X be a point of $|X-T| < \frac{\rho}{3}$, and Y the symmetric

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2. Here slightly modified.

point with respect to T , that is $Y = 2T - X$. Let $x=f(X)$, $y=f(Y)$. Then $|x-t| = |X-T| < \frac{\rho}{3}$,

$$|y-t| = |Y-T| < \frac{\rho}{3}, \quad \left|\frac{x+y}{2} - t\right| < \frac{\rho}{3}, \quad \left|\frac{x-y}{2}\right| < \frac{2}{3}\rho .$$

Thus the ball with diameter x,y is contained in the ball $|z-t| < \rho$. Hence the sets $\sum_n$ constructed successively from the points X,Y will have as images exactly the corresponding sets constructed from x and y. It follows that $f(T) = \frac{1}{2}(x+y)$.

Let now U and V be points with $|U-T| < \frac{\rho}{4}$, $|V-T| < \frac{\rho}{4}$. Then the ball of radius $\frac{3}{2}|V-U|$ about the midpoint of U and V lies in R. It follows that

$$(5^*.3) \qquad f\left(\frac{U+V}{2}\right) = \frac{f(U)+f(V)}{2} .$$

This relation holds for any U,V in a $\frac{\rho}{4}$-neighbourhood of T. By continuity of f then more generally

$$(5^*.4) \quad f(\lambda U+\mu V) = \lambda f(U)+\mu f(V) \text{ for } \lambda+\mu = 1, \ \lambda \geqq 0, \ \mu \geqq 0 .$$

One easily convinces oneself that f in a neighbourhood of T coincides with an affine mapping g(X) defined in the whole space. Indeed the mapping g(Z) can be defined for all Z by

$$g(T) = f(T)$$

and

$$g(Z) = f(T) + \frac{8}{\rho}|Z-T|\left(f\left(T + \frac{\rho}{8|Z-T|}(Z-T)\right)-f(T)\right) \text{ for } Z \neq T.$$

Here $g(Z) = f(Z)$ for $|Z-T| \leqq \frac{\rho}{8}$ by (5*.4). The mapping g is rigid everywhere, since it is affine and coincides with a rigid mapping in a neighbourhood of T.

We see that f in a neighbourhood of any point T of R agrees with an affine rigid mapping g. Now two affine mappings that coincide in an open set coincide everywhere. It follows by continuation that f coincides with the same affine mapping g in a neighbourhood of any point T' of R that can be joined to T inside R by a polygonal arc with a finite number of vertices. Since, by assumption, R is an open connected set in the Banach space B it is possible to join any two points of R by such a polygon. It follows that f coincides throughout R with the same affine rigid mapping.

6. Mappings of ellipsoids.

Theorem II gives no lower bound for $|f(Y)-f(X)|$ when Y and X are points of the domain of f whose mutual distance is large compared to their distance from the boundary of the domain. There will then be no ball in the domain containing both X and Y. In some cases of interest one can then still make use of the following theorem:[1]

Theorem IV.

Define in Banach space B the ellipsoid of revolution E^k_{XY} with foci X,Y and eccentricity k by

$$(6.1) \qquad E^k_{XY} = (Z\colon\ |X-X| + |Z-Y| < \tfrac{1}{k}|Y-X|)$$

where $k < 1$. Let f be (m,M)-isometric in R. Then

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1. In case B is a Hilbert space theorem II can be deduced from theorem IV by elementary geometry, as will be shown in the sequel.

(6.2) $$m|Y-X| \leqq |f(Y)-f(X) \leqq M|Y-X|$$

for any two points X,Y for which the ellipsoid E_{XY}^{k} with foci X,Y and eccentricity

(6.3) $$k = \frac{m}{M}$$

is contained in R.[1]

Proof:

Since the ellipsoid E_{XY}^{k} contains the convex hull of its foci X,Y we conclude immediately from lemma I that

$$|f(Y)-f(X)| \leqq M|Y-X| .$$

It remains to prove

(6.4) $$m|Y-X| \leqq |f(Y)-f(X)|$$

under the assumption that R contains the ellipsoid E_{XY}^{k} for $k = m/M$. It is sufficient to prove (6.4) under the stronger assumption that R contains some ellipsoid E_{XY}^{k} with $k < m/M$. Indeed, if R contains E_{XY}^{k} and X',Y' are any points on the open segment with endpoints X,Y then R contains some $E_{X'Y'}^{k'}$ with $k' < k$. It follows then from the weaker statement that

$$m|Y'-X'| \leqq |f(Y')-f(X')|$$

for any X',Y' between X and Y. From the continuity of f equation (6.4) would follow.

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1. Proofs of this theorem for the case of euclidean spaces B and b were given by the author in [4], [5]. The proof had to be modified for general Banach spaces for lack of compactness.

Assume then that $E^k_{XY} \subset R$ where k is some value with

$$k < \frac{m}{M} \tag{6.5}$$

Then

$$|Z-X| + |Z-Y| \geqq \frac{1}{k}|Y-X| > \frac{M}{m}|Y-X| \quad \text{for } Z \notin R\ . \tag{6.6}$$

Consider now the points

$$X_\lambda = (1-\lambda)X+\lambda Y \quad \text{for} \quad 0 \leqq \lambda \leqq 1\ .$$

Then $X_o = X$, $X_1 = Y$. Let for $0 \leqq \mu \leqq 1$ the arc $\sum_\mu$ be defined by

$$\sum_\mu = (X_\lambda\colon\ 0 \leqq \lambda \leqq \mu)$$

Put $x_\lambda = f(X_\lambda)$. We consider the star s_{X_o} and the inverse $f^{-1}_{X_o}$ defined in that star. For simplicity we just write s for the star and f^{-1} for the inverse. If $f(\sum_\mu)$ lies completely in the star s then by p. 3.

$$f^{-1}(x_\mu) = X_\mu \tag{6.7}$$

Since f^{-1} is (M^{-1},m^{-1})-isometric in s it follows then that

$$|f^{-1}(x_\mu)-f^{-1}(x_o)| \leqq m^{-1}|x_\mu-x_o|\ ,$$

that is

$$m|X_\mu-X_o| \leqq |f(X_\mu)-f(X_o)|\ .$$

In particular if $f(\sum_1) \subset s$ we would have for $\mu = 1$

$$m|Y-X| \leqq |f(Y)-f(X)|$$

which is just the statement to be proved.

Hence we only have to consider the case where $f(\sum_1) \not\subset s$, that is the case where there exist λ with $0 \leqq \lambda \leqq 1$ for which $x_\lambda \notin s$. (See Fig. 1.) There will be a smallest such λ, say $\lambda = \nu$, since s is open. Equation (6.7) will hold for $0 \leqq \mu < \nu$. Since x_ν does not belong to s it must not be possible to continue f^{-1} all the way along the ray from x_o to x_ν. There exists then a smallest positive $\alpha \leqq 1$ such that the point $z = x_o + \alpha(x_\nu - x_o)$ does not belong to s. Moreover by Cauchy's test

$$(6.8) \qquad \lim_{\lambda \to \alpha-} f^{-1}(x_o + \lambda(x_\nu - x_o)) = Z$$

exists and is a boundary point of R.

For $\mu < \nu$ and $0 \leqq \lambda \leqq 1$ the points $x_o + \lambda(x_\mu - x_o))$ lie in s, and consequently

$$(6.9a) \qquad |f^{-1}(x_o + \lambda(x_\mu - x_o)) - f^{-1}(x_\mu)| \leqq m^{-1}(1-\lambda)|x_\mu - x_o|$$

$$(6.9b) \qquad |f^{-1}(x_o + \lambda(x_\mu - x_o)) - f^{-1}(x_o)| \leqq m^{-1}\lambda|x_\mu - x_o| .$$

Here $f^{-1}(x_\mu) = X_\mu$, $f^{-1}(x_o) = X_o$. For $\lambda < \alpha$ the point $x_o + \lambda(x_\nu - x_o)$ belongs to s and f^{-1} is continuous at that point. Letting μ tend to ν we find that for $0 < \lambda < \alpha$

$$(6.10a) \qquad |f^{-1}(x_o + \lambda(x_\nu - x_o)) - X_\nu| \leqq m^{-1}(1-\lambda)|x_\nu - x_o|$$

$$(6.10b) \qquad |f^{-1}(x_o + \lambda(x_\nu - x_o)) - X_o| \leqq m^{-1}\lambda|x_\nu - x_o| .$$

Hence for $\lambda \to \alpha$ by (6.8)

$$(6.11) \qquad |Z - X_\nu| \leqq m^{-1}(1-\alpha)|x_\nu - x_o| , \qquad |Z - X| \leqq m^{-1}\alpha|x_\nu - x_o| .$$

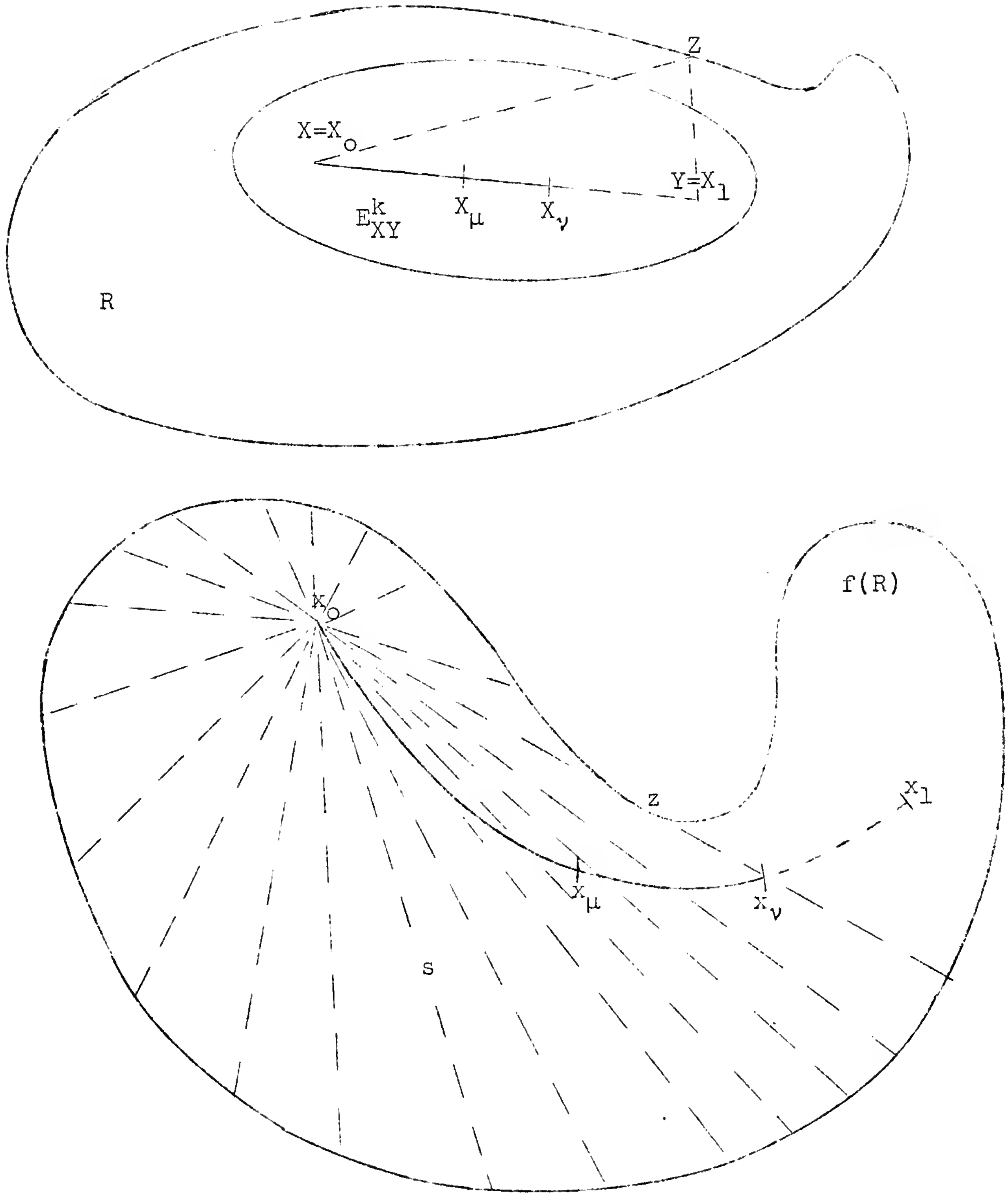


Figure 1.

Hence

$$|Z-X|+|Z-X_\nu| \leqq m^{-1}|x_\nu-x_o| = m^{-1}|f(X_\nu)-f(X)| \leqq Mm^{-1}|X_\nu-X| .$$

Consequently

$$|Z-X|+|Z-Y| \leqq |Z-X|+|Z-X_\nu|+|X_\nu-Y| \leqq Mm^{-1}|X_\nu-X|+|X_\nu-Y|$$

$$\leqq Mm^{-1}(|X_\nu-X|+|X_\nu-Y|) = Mm^{-1}|Y-X|$$

which contradicts (6.6).

II. Mappings between Hilbert spaces.

7. Elliptical hulls.

In all that follows we shall make the assumption that the spaces B and b are Hilbert spaces. The scalar product of two elements X,Y of the same space will be denoted by $X \cdot Y$, so that $|X|^2 = X \cdot X$. Since all finite-dimensional subspaces of a Hilbert space are euclidean, geometry in such a space agrees perfectly with euclidean intuition. In particular ellipsoids of revolution E^k_{XY} look like their euclidean counterparts in 3-space.

Definition:

Given a set S in the Hilbert space B and a number $k < 1$ we define the E^k-hull of S as the union of all ellipsoids with eccentricity k and foci in S .

For a set consisting of two points X,Y the E^k-hull is just the ellipoid E^k_{XY} .

<u>Lemma II: The E^k-hull of a convex set S of diameter d is contained in the δd-neighbourhood of S, where</u>

$$\delta = \frac{1}{2}\sqrt{\frac{1}{k^2} - 1} \tag{7.1}$$

Proof: Let Z be a point of the E^k-hull of the convex set S. Then there exist points X,Y in S such that $Z \in E^k_{XY}$. The δd-neighbourhood of S includes, because of the convexity of S, the δd-neighbourhood of the segment with endpoints X,Y. It is sufficient to prove that Z belongs to the latter neighbourhood. The two-plane through X,Y,Z intersects E^k_{XY} in the area bounded by the ordinary ellipse with foci X,Y and eccentricity k. Here $2\delta d$ is just the minor axis of the ellipse. It is sufficient to prove that the endpoints of the minor axis of an ellipse are the points on the ellipse farthest away from the segment that has the foci as endpoints. (See Fig. 2.) This is easily verified.

<u>Lemma III: The E^k-hull of the ball</u> $|X-X^o| < \rho$ <u>is the ball</u> $|X-X^o| < \frac{1}{k}\rho$.

Proof:

We first show that the E^k-hull of the ball $|X-X^o| < \rho$ is contained in the ball $|X-X_o| < \frac{1}{k}\rho$. Let X and Y be points satisfying $|X-X^o| < \rho$, $|Y-X^o| < \rho$. We have to prove that for $Z \in E^k_{XY}$ the inequality

$$|Z-X^o| < \frac{1}{k}\rho \tag{7.2}$$

is satisfied. Let, without restriction of generality,

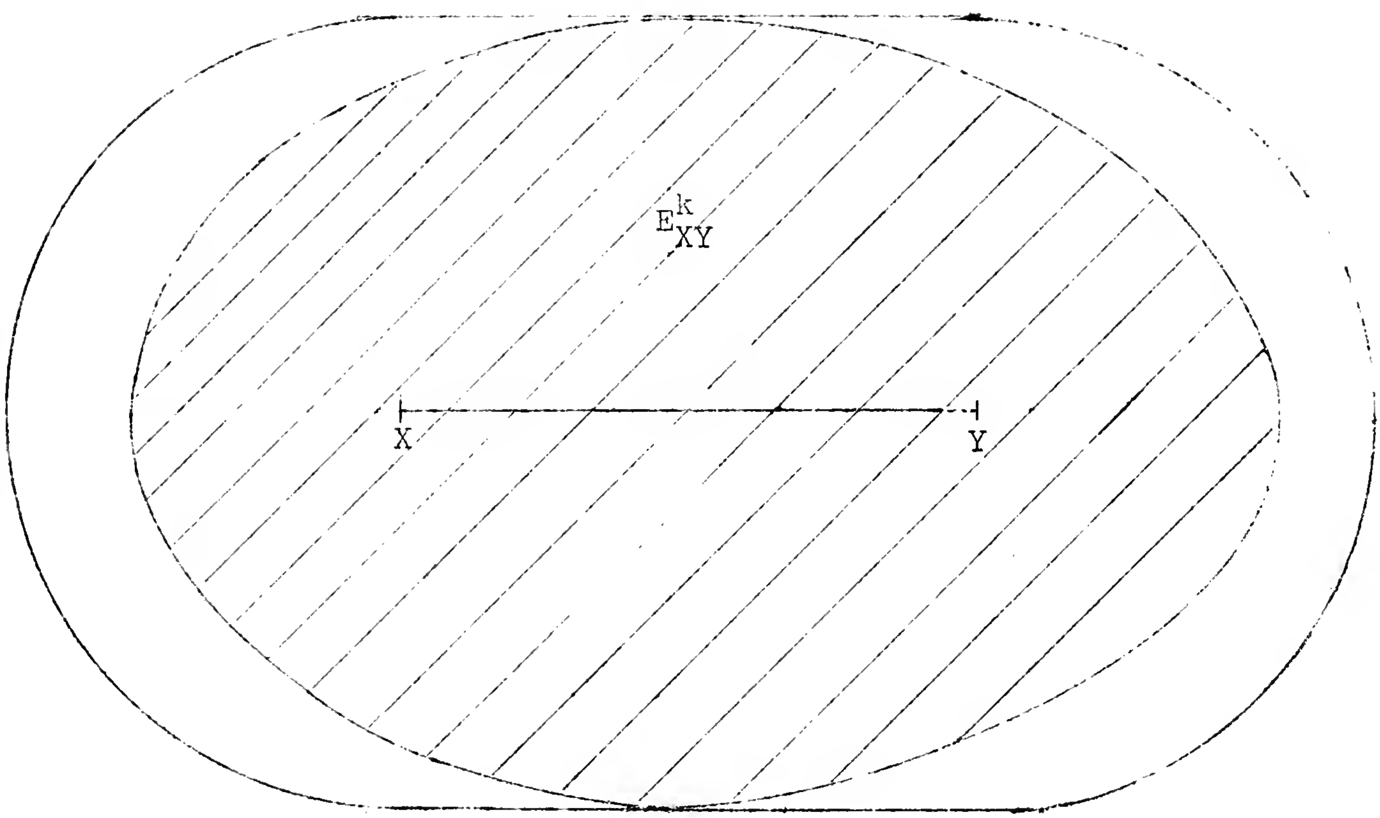


Figure 2.

$|Y-X^o| \leqq |X-X^o|$. There will exist a point Y' with $|Y'-X^o| = |X-X^o|$ such that Y lies on the segment with endpoints X and Y'. Obviously $E^k_{XY} \in E^k_{XY'}$. It is sufficient to prove (7.2) for $Z \in E^k_{XY'}$. Intersecting with the 3-plane through the four points X, Y', X^o, Z we only have to prove the following proposition: In euclidean 3-space let X and Y' be points with

$$(7.3) \qquad |X-X^o| = |Y-X^o| = \lambda < \rho \; .$$

Let Z be a point of the ellipsoid with foci X, Y' and eccentricity k. Then

$$(7.4) \qquad |Z-X^o| < \frac{1}{k}\,\rho \; .$$

Inequality (7.4) will follow if we can prove

$$(7.5) \qquad |Z-X^o| \leqq \frac{1}{k}\,\lambda \; .$$

There is now a smallest sphere about X^o which contains the ellipsoid with foci X, Y' and eccentricity k. Let μ be the radius of that sphere. We want to show that $\mu \leqq \frac{1}{k}\lambda$. The sphere of radius μ and center X^o will touch the boundary of the ellipsoid at a point T. Since the normals of ellipsoid and sphere at T coincide, we see that the four points X, Y', T, X^o lie in the same two dimensional plane. In this plane we have (see Fig. 3) an ellipse with foci X, Y' and eccentricity k touching a circle of radius μ and center X^o from the inside. Moreover X^o lies on the extended minor axis of the ellipse and has distance λ from the two foci. Elementary geometry shows that then actually $\mu = \frac{1}{k}\lambda$.

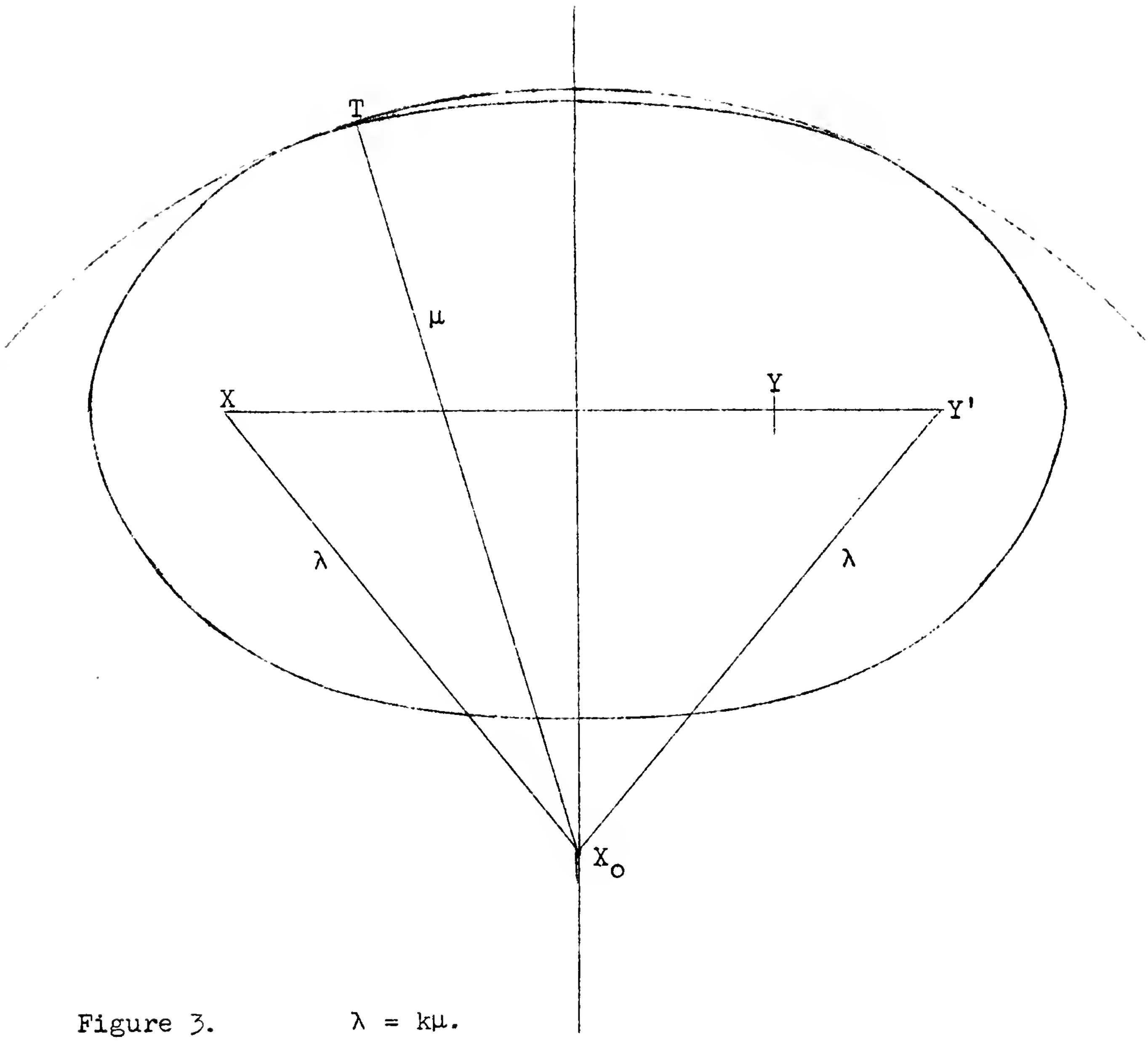


Figure 3. $\lambda = k\mu$.

This proves that the E^k-hull of the ball $|X-X^o| < \rho$ is contained in the ball $|X-X^o| < \frac{1}{k}\rho$. We still have to show that every point Z with $|Z-X^o| < \frac{1}{k}\rho$ belongs to some E^k_{XY} with $|X-X^o| < \rho$, $|Y-X^o| < \rho$. Take for X,Y simply the points

$$X = X^o + \sigma(Z-X^o) \ , \quad Y = X^o - \sigma(Z-X^o)$$

where σ-k is positive and sufficiently small.

8. Applications to quasi-isometric mappings.

Using the notion of elliptical hull we can trivially reformulate theorem IV of p. 15 in the following way:

Theorem IV'.

Let f(X) be an (m,M)-isometric mapping of an open set R in the space B. Let S be a subset of R with the property that for

$$(8.1) \qquad k = \frac{m}{M}$$

the E^k-hull of S lies in R. Then the mapping f is (m,M)-rigid in S.

If now B is a Hilbert space we draw from lemmas II and III, p.20, immediately the following consequences:[1]

Corollary III:

Let R be a convex set of diameter d and f(X) an (m,M)-isometric mapping of R. Let S be the convex subset of R consisting of the points that have a distance from the boundary

1. That b also is a Hilbert-space is not used here.

<u>of R exceeding the value</u>

(8.2) $$\frac{1}{2}\sqrt{\left(\frac{M}{m}\right)^2 - 1}\ d \quad .$$

<u>Then f is (m,M)-rigid in S.</u> (Observe that for M/m close to the value 1 the set S is almost the whole of R.)

<u>Corollary IV:</u>

<u>An (m,M)-isometric mapping of a ball of radius</u> ρ <u>is (m,M)-rigid in the concentric ball of radius</u>

(8.3) $$\frac{m}{M}\rho \quad .$$

(This is of course the previously proved theorem II, p. 10.)

Quasi-isometric mappings are one-one in the small. The question arises under what circumstances one can be sure that they are also one-one in the large. It is intuitively obvious that such mappings of a domain R will be more likely to be 1-1 if R is not too longstretched and M/m is not too large. We will first occupy ourselves with the case where R is a ball.

An (m,M)-mapping of the whole space is necessarily one-one, by Corollary I. But if the domain of the mapping is only a ball of finite radius the mapping need not be one-one if M/m is sufficiently large. Thus the conformal mapping $z = e^Z$ is not one-one in the disk $|Z| < \pi(1+\varepsilon)$ where ε is any positive number. In that disk the mapping is (m,M)-isometric with $m = e^{-\pi(1+\varepsilon)}$, $M = e^{\pi(1+\varepsilon)}$, and hence

$$\frac{M}{m} = e^{2\pi(1+\varepsilon)} \quad .$$

Even for smaller values of M/m the mapping does not have to be one-one in a ball. Consider in the plane the mapping of the right half-plane that takes a point with polar coordinates $R,\emptyset$ (where $R > 0$, $|\emptyset| < \pi/2$) into the point with polar coordinates r,ϕ, where $r = R$, $\phi = (2+\varepsilon)\emptyset$, and where ε is any positive number. It is clear that for this mapping of the half-plane $m = 1$, $M = 2+\varepsilon$, and hence

$$\frac{M}{m} = 2+\varepsilon \tag{8.4}$$

It is also clear that this mapping of the half-plane is not 1-1. So there are two points in the half-plane with the same image. We can always find a circular disk in the half-plane that contains the two points. We have then a quasi-isometric mapping of the disk with (8.4) that is not 1-1.

Theorem V.

An (m,M)-isometric mapping of a ball in Hilbert-space B is 1-1 if

$$\frac{M}{m} < \sqrt{\frac{1+\sqrt{5}}{2}} = 1.27\ldots \tag{8.5}$$

More precisely, when (8.5) is satisfied, the mapping is (μ,M)-rigid in the ball with

$$\mu = \frac{m^2-M\sqrt{M^2-m^2}}{m+\sqrt{M^2-m^2}} . \tag{8.6}$$

Proof: Let f be the (m,M)-isometric mapping of a ball of radius ρ in Hilbert-space. Without restriction of generality we assume that the center of the ball is at the origin, so that

the ball is given by $|X| < \rho$. Let X, Y be any two distinct points of the ball. Then for $0 \leqq \theta \leqq 1$ by Lemma I

$$\begin{aligned} |f(Y)-f(X)| &\geqq |f(\theta Y)-f(\theta X)|-|f(Y)-f(\theta Y)|-|f(X)-f(\theta X)| \\ &\geqq |f(\theta Y)-f(\theta X)|-M(1-\theta)(|Y|+|X|) \\ &\geqq |f(\theta Y)-f(\theta X)|-2M(1-\theta)\rho \, . \end{aligned}$$

Let $d = \theta|Y-X|$ be the distance of the points θX and θY, and let $k = m/M$. We choose θ in such a way that both the points X and Y have distance at least

$$\frac{1}{2}\sqrt{\frac{1}{k^2} - 1} \quad d \tag{8.7}$$

from the boundary of the ball. (See Fig. 4.) Since the ball is convex every point on the segment with endpoints θX, θY will then at least have that distance from the boundary of the ball. It follows then from Corollary III, p.22, that f is (m,M)-rigid on the segment with endpoints θX and θY. In particular it follows that

$$|f(\theta Y)-f(\theta X)| \geqq m\theta|Y-X| \, ,$$

and hence that

$$|f(Y)-f(X)| \geqq m\theta|Y-X| - 2M(1-\theta)\rho \, . \tag{8.8}$$

The distance of the points θX and θY from the boundary of the ball is at least $(1-\theta)\rho$. Thus the points θX, θY will have a distance from the boundary at least equal to the expression (8.7), and (8.8) will hold, if

$$(1-\theta)\rho > \frac{1}{2}\sqrt{\frac{1}{k^2} - 1} \quad \theta|Y-X| \, .$$

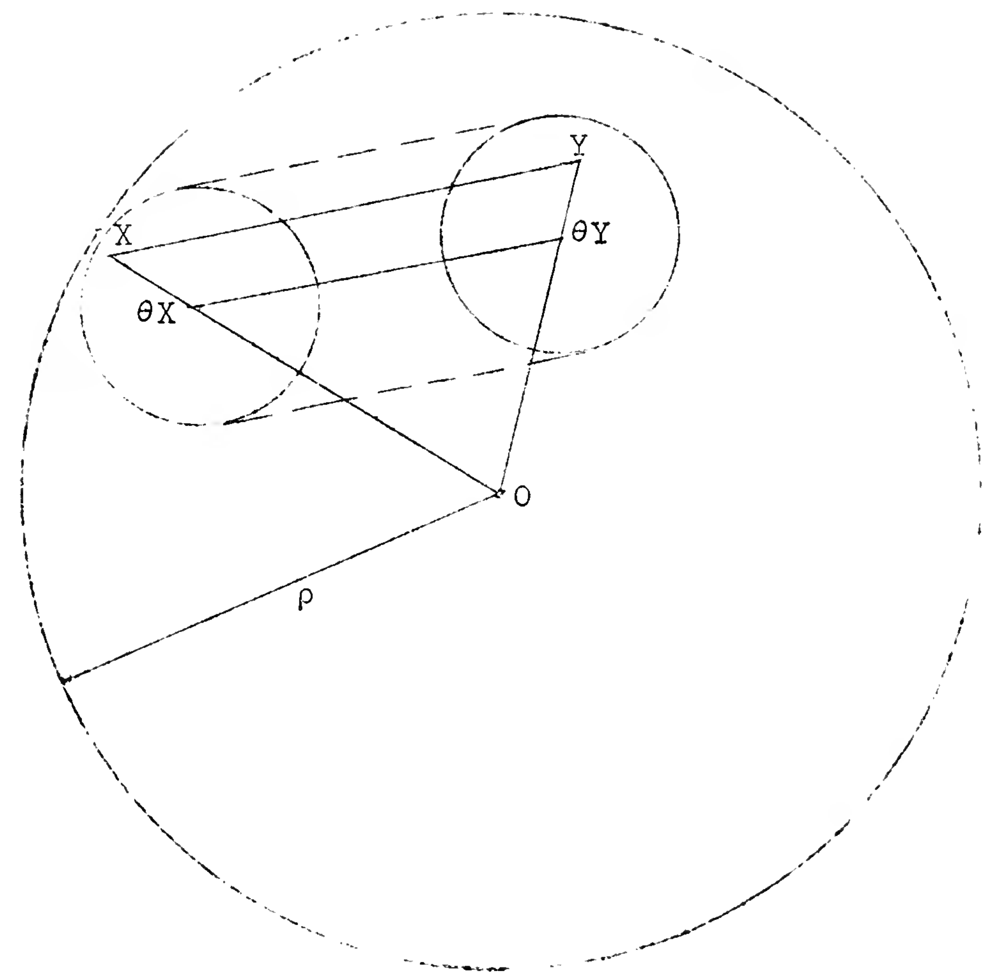


Figure 4.

that is if

(8.9) $$\theta \leqq \frac{\rho}{\rho + \frac{1}{2}(k^{-2}-1)^{1/2}|Y-X|} .$$

Then by (8.8)

$$|f(Y)-f(X)| \geqq \mu|Y-X|$$

provided that

$$m\theta|Y-X|-2M(1-\theta)\rho \geqq \mu|Y-X|$$

that is

(8.10) $$\theta \geqq \frac{\mu|Y-X|+2M\rho}{m|Y-X|+2M\rho} .$$

We can always find a θ with $0 \leqq \theta \leqq 1$ satisfying both (8.9) and (8.10) if

$$\mu\rho + \frac{\mu}{2}(k^{-2}-1)^{1/2}\ |Y-X| \leqq m\rho - M\rho\ (k^{-2}-1)^{1/2} .$$

Since $|Y-X| < 2\rho$ this is certainly the case when

(8.11) $$\mu = \frac{m-M(k^{-2}-1)^{1/2}}{1+(k^{-2}-1)^{1/2}} .$$

Since here $k = m/M$ this value of μ reduces to the one given by (8.6). Here μ is positive, and consequently the mapping is one-one, when (8.5) holds.

9. Numerical improvement of the preceding results.

Obviously there exists a universal constant γ such that an (m,M)-mapping of a ball in Hilbert-space into a Hilbert-space is one-one, when $M/m < \gamma$, but need not be one-one when $M/m > \gamma$. Here by (8.4), (8.5)

$$1.27 < \gamma \leqq 2 .$$

We can narrow down the bounds on γ, and incidentally sharpen theorem II by proving theorem IV' with k replaced by a larger value than m/M. In contrast to the developments in section 8 we shall here make use of the assumption that not only B but also b are Hilbertian.

Theorem: VI:

Let f be an (m,M)-mapping of an open set R in Hilbert-space B into Hilbert-space b. Let S be a subset of R with the property that for

$$k = \sqrt{\frac{2}{1+M^2m^{-2}}} \tag{9.1}$$

the E^k-hull of S lies in R. Then the mapping f is (m,M)-rigid in S.

Proof: The proof is a modification of the proof of theorem IV. Again it is sufficient to prove that

$$m|Y-X| \leqq |f(Y)-f(X)| \tag{9.2}$$

when X and Y are any two points for which $E^k_{XY} \subset R$ for some number k with

$$k < \sqrt{\frac{2}{1+M^2m^{-2}}} \quad . \tag{9.2a}$$

Let again $X_\lambda = (1-\lambda)X+\lambda Y$ and $x_\lambda = f(X_\lambda)$. We now introduce for

$$0 \leqq \alpha \leqq \beta \leqq 1 \tag{9.3}$$

the arcs $\sum_{\alpha\beta}$ by

$$\sum\nolimits_{\alpha\beta} = (X_\lambda: \quad \alpha \leqq \lambda \leqq \beta) \ .$$

As before we have $f_{X_\alpha}^{-1}(x_\beta) = X_\beta$ and

$$(9.4) \qquad m|X_\beta - X_\alpha| \leqq |f(X_\beta) - f(X_\alpha)|$$

in the case where the whole arc $f(\sum_{\alpha\beta})$ lies in the star s_{X_α}. If now (9.2) were wrong there must be α,β satisfying (9.3) for which the arc $f(\sum_{\alpha\beta})$ is not completely contained in s_{X_α}. Let $\mu = \inf (\beta-\alpha)$ for the α,β with these properties. It is easily seen that there exist α,β with $\beta-\alpha=\mu$, satisfying (9.3) and such that s_{X_α} does not contain $\sum_{\alpha\beta}$. For there are certainly sequences of values α,β satisfying (9.3) and with $f(\sum_{\alpha\beta}) \not\subset s_{X_\alpha}$ for which $\beta-\alpha \to \mu$. For suitable subsequences the α,β have limits, again denoted by α,β for which $\beta-\alpha=\mu$. For the limits we cannot have $f(\sum_{\alpha\beta}) \not\subset s_{X_\alpha}$ since this would then hold also for all neighbouring α,β because the stars are open sets.

Thus the incorrectness of (9.2) leads to the existence of points X_α, X_β in the closed segment with endpoints X,Y, for which the arc $f(\sum_{\alpha\beta})$ does not belong to s_{X_α}. However $f(X_\lambda) \subset s_{X_\gamma}$ for $\alpha \leqq \gamma \leqq \lambda < \beta$. Moreover the ellipsoid with foci X_α, X_β and eccentricity k given by (9.2a) also lies in R. We shall prove that that is impossible. Without restriction of generality we can assume that $\alpha = 0$, $\beta = 1$, that is that X_α and X_β are the original points X,Y. We arrive then at points X,Y with $E_{XY}^k \subset R$, where k satisfies (9.2a) and which are such that

$$(9.5) \qquad x_\lambda = f(X_\lambda) \in s_{X_\gamma} \quad \text{for} \quad 0 \leqq \gamma \leqq \lambda < 1 ,$$

(9.6) $$x_1 = f(Y) \notin s_{X_0}$$

Let us denote by ρ_λ for $0 \leqq \lambda \leqq 1$ the distance of the point X_λ from the boundary of R. By theorem I the star s_{X_λ} includes the ball b_λ (see Fig. 5) of radius $m\,\rho_\lambda$ and center x_λ. In that ball the inverse $f_{X_\lambda}^{-1}(x)$ is surely defined. Let the union of all these balls be

$$r = \bigcup_{0 \leqq \lambda < 1} b_\lambda \ .$$

If x is a point common to two of these balls, say $x \in b_\lambda$, $x \in b_\mu$ where $0 \leqq \lambda < \mu < 1$, then $f_{X_\lambda}^{-1}(x) = f_{X_\mu}^{-1}(x)$. For $f(\sum_{\lambda\mu}) \subset s_{X_\lambda}$ and thus

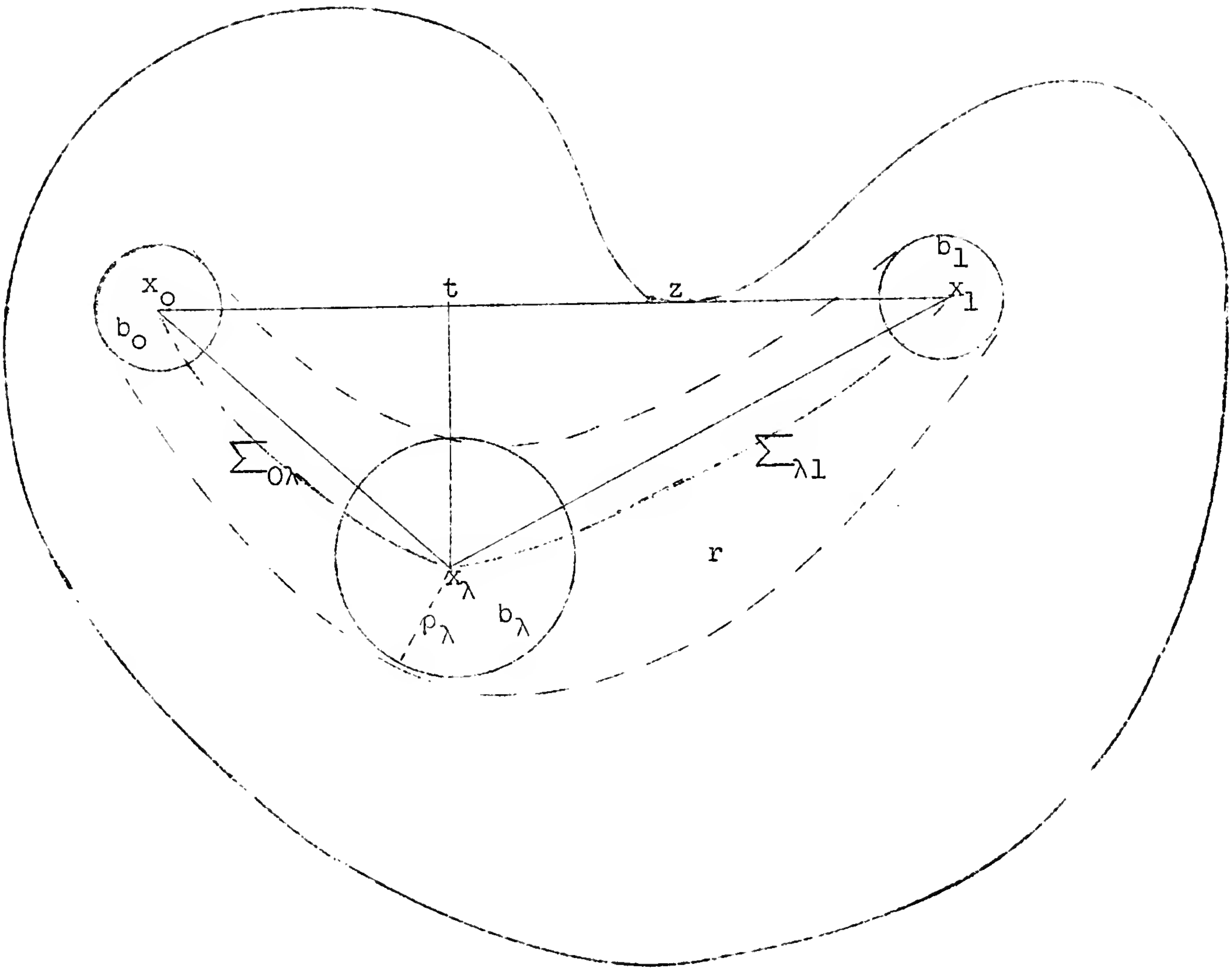
$$f_{X_\lambda}^{-1}(x_\mu) = X_\mu = f_{X_\mu}^{-1}(x_\mu) \ .$$

$f_{X_\lambda}^{-1}$ and $f_{X_\mu}^{-1}$ are both defined and agree at x_μ; they agree then at any point that can be joined to x_μ by a segment on which both are defined; the point x common to b_λ and b_μ was just such a point. Thus the various functions $f_{X_\lambda}^{-1}(x)$ for $0 \leqq \lambda < 1$ uniquely define an inverse $f^{-1}(x)$ in the set r, where

$$f^{-1}(x_0) = f_{X_0}^{-1}(x_0) = X_0 \ .$$

As indicated by (9.6) there exists an α with $0 < \alpha \leqq 1$ such that

$$Z = \lim_{\lambda \to \alpha-} f_{X_0}^{-1}(x_0 + \lambda(x_1 - x_0))$$



x_o
b_o
t
z
b_1
x_1
$\Sigma_{0\lambda}$
$\Sigma_{\lambda 1}$
r
x_λ
ρ_λ
b_λ

Figure 5.

exists and is a boundary point of R. The whole closed segment with endpoints x_o and $z = x_o+\alpha(x_1-x_o)$ cannot belong to r; otherwise f^{-1} would be defined along the segment and constitute a continuation of $f_{X_o}^{-1}$ along the segment, including the point $x_o+\alpha(x_1-x_o)$; but by assumption such a continuation does not exist. Thus there exists a μ with $0 < \mu \leqq \alpha$ such that

$$t = x_o + \mu(x_1-x_o) \notin r ,$$

that is such that

$$|x_o + \mu(x_1-x_o) - x_\lambda| \geqq m\rho_\lambda \tag{9.7}$$

for $0 \leqq \lambda < 1$. The same inequality still holds for $\lambda = 1$ since both x_λ and ρ_λ depend continuously on λ.

We have, as in (6.11),

$$|Z-X_o| \leqq m^{-1}\alpha|x_1-x_o| , \quad |Z-X_1| \leqq m^{-1}(1-\alpha)|x_1-x_o| .$$

Consequently

$$\begin{aligned} \tfrac{1}{K}|Y-X| &\leqq |Z-X|+|Z-Y| = |Z-X_o|+|Z-X_1| \\ &\leqq m^{-1}|x_1-x_o| \\ &= m^{-1}(\mu|x_1-x_o|+(1-\mu)|x_1-x_o|) \end{aligned} \tag{9.8}$$

Now, for the first time, we make use of the fact that the space b is a Hilbert-space. The expression

$$(x_o+\mu(x_1-x_o)-x_\lambda) \cdot (x_1-x_o) = \phi_\lambda$$

is positive for $\lambda = 0$ and non-positive for $\lambda = 1$. There exists

then a λ such that $0 < \lambda \leqq 1$ and $\phi_\lambda = 0$. For that λ, by the theorem of Pythagoras,

$$|x_o+\mu(x_1-x_o) - x_\lambda|^2 = |x_\lambda-x_o|^2 - \mu^2|x_1-x_o|^2$$
$$= |x_\lambda-x_1|^2 - (1-\mu)^2|x_1-x_o|^2 .$$

It follows from (9.7), (9.8) that

$$\frac{1}{k}|Y-X| \leqq m^{-1}\left(\sqrt{|x_\lambda-x_o|^2 - m^2\rho_\lambda^2} + \sqrt{|x_\lambda-x_1|^2 - m^2\rho_\lambda^2}\right)$$

Using Lemma 1, p. 7, we have then

$$(9.9) \quad \frac{1}{k}|Y-X| \leqq m^{-1}\left(\sqrt{M^2\lambda^2|Y-X|^2-m^2\rho_\lambda^2} + \sqrt{M^2(1-\lambda)^2|Y-X|^2-m^2\rho_\lambda^2}\right)$$

An elementary computation shows that for

$$X_\lambda = (1-\lambda)X+\lambda Y \quad , \quad 0 \leqq \lambda \leqq 1$$

the ball of radius

$$\sqrt{(k^{-2}-1)\lambda(1-\lambda)} \quad |Y-X|$$

is contained in the ellipsoid E^k_{XY}. Hence

$$\rho_\lambda \geqq \sqrt{(k^{-2}-1)\lambda(1-\lambda)} \quad |Y-X| .$$

It follows from (9.9) that there exists a λ in the interval $0 \leqq \lambda \leqq 1$ for which

$$\frac{1}{k} \leqq m^{-1}\left(\sqrt{M^2\lambda^2-m^2(k^{-2}-1)\lambda(1-\lambda)} + \sqrt{M^2(1-\lambda)^2-m^2(k^{-2}-1)\lambda(1-\lambda)}\right)$$

The right-hand side, which is a concave function of λ and even

in $\lambda - \frac{1}{2}$, reaches its single maximum at $\lambda = \frac{1}{2}$. It follows that

$$\frac{1}{k} \leqq m^{-1} \sqrt{M^2-m^2(k^{-2}-1)}\,.$$

This however contradicts (9.2a).

Using lemma III, p.22, we immediately obtain from the theorem just proved the following improvement on theorem II, p.10:

Corollary VI: Let f be an (m,M)-isometric mapping of a ball of radius ρ in Hilbert-space B into Hilbert-space b. Then f is (m,M)-rigid in the concentric ball of radius $k\rho$, where k is given by

$$k = \sqrt{\frac{2}{1+M^2m^{-2}}} \quad . \tag{9.10}$$

Using (8.11) with the value of k given by (9.2) we have the following improvement on theorem V, p.24:

Corollary VII:

An (m,M)-isometric mapping of a ball in Hilbert-space B into a Hilbert-space b is 1-1 if

$$\frac{M}{m} < \sqrt{2} = 1.41\ldots \quad . \tag{9.11}$$

More precisely, when (9.11) is satisfied, the mapping is (μ,M)-rigid in the whole ball with

$$\mu = \frac{m^2 - M\sqrt{\frac{M^2-m^2}{2}}}{m + \sqrt{\frac{M^2-m^2}{2}}} \tag{9.12}$$

Thus the bounds for the universal constant γ defined on p. 26.

have been narrowed down to

$$1.41 < \gamma \leq 2 .$$

As a special case we have:

Corollary VIII:

A conformal mapping of a circular disk given by an analytic function $F(Z) = F(X+iY)$ will be schlicht, if there exist constants m, M such that in the disk everywhere

$$m \leq |F'(Z)| \leq M$$

and such that

$$\frac{M}{m} < \sqrt{2} .$$

10. About sets that are quasi-isometrically equivalent to balls.

The question when two sets can be mapped into one another by a quasi-isometric mapping suggests itself naturally. We consider in particular open sets R in Hilbert space B for which there exists quasi-isometric mappings which map R one-one onto a ball in B.[1]

Each quasi-isometric mapping f with domain R has a greatest m and smallest M for which it is (m,M)-isometric in R. We call the quantity

$$k = \sqrt{1 - \frac{m^2}{M^2}} \tag{10.1}$$

1. Notice that the mappings are required to be one-one for the whole of R, and also to have their range in the same Hilbert space B that contains the domain of the mapping.

the <u>eccentricity</u> of the mapping f . Multiplying the mapping function f by a constant does not change the eccentricity of the mapping. We can then always choose the constant in such a way that for the resulting "normalized" mapping the relation $mM = 1$ holds. The quantity k then measures the non-isometric character of the mapping. Introducing the related quantity

$$(10.2) \qquad \varepsilon = (1-k^2)^{-1/4} = \sqrt{\frac{M}{m}} - 1$$

we have for the mappings normalized by $mM = 1$ the relations

$$(10.3) \qquad m = \frac{1}{1+\varepsilon} \quad , \qquad M = 1+\varepsilon \quad ,$$

and ε can be considered as the <u>maximum strain</u> of the mapping.

Let now R be an open set in Hilbert-space B. Assume that there exist one-one quasi-isometric mappings of R onto a ball in B. We define the <u>eccentricity of the set</u> R as the greatest lower bound of the eccentricities of all such mappings. The eccentricity of R measures in a sense the least strain sure to be generated somewhere in deforming R into a ball. Open sets R that can be mapped one-one and quasi-isometrically into balls will be called, for short, <u>spheroids</u>. A spheroid then is a set that can be deformed into a sphere (or rather a "ball") without causing infinite strains. Spheroids have an eccentricity k for which $0 \leqq k < 1$.

The definition of eccentricity of a set given here is reasonable in view of the following two lemmas:

Lemma IV:

A set of eccentricity 0 is a ball.

Lemma V:

For an ellipsoid of revolution E^k_{XY} the eccentricity, as defined just now, coincides with the eccentricity k in the ordinary meaning of the word.

Proof of lemma IV.

Consider a 1-1 quasi-isometric mapping f of eccentricity k of the open set R onto a ball. We can always normalize the mapping in such a way that the center X_o of the ball coincides with its pre-image and such that $mM = 1$ for the mapping. Let the ball be $|X-X_o| < \rho$. The inverse f^{-1} also is a 1-1 quasi-isometric mapping of the ball onto R, and with the same m,M. By theorem I, p.9, and lemma I, p.7, the set R will contain the ball $|X-X_o| < m\rho$ and be contained in the ball $|X-X_o| < M\rho$, where m,M are related to k by (10.2). For a different mapping f' of eccentricity k' of R onto a ball we obtain similarly after normalization that R is contained in the ball $|X-X'_o| < M'\rho'$ and contains the ball $|X-X'_o| < m'\rho'$. It follows that $|X_o-X'_o| + m'\rho' \leq M\rho$, $|X_o-X'_o| + m\rho < M'\rho'$. Hence

$$|X_o-X'_o| \leq \frac{MM'-mm'}{m}\rho' .$$

It follows that for $k \to 0$ the centers X_o and radii ρ converge, and that R is a ball.

Proof of lemma V.

The points X,Y are the foci of the ellipsoid E^k_{XY}. We can

use these points to introduce "cylindrical coordinates". For any point Z we put

$$Z = \frac{X+Y}{2} + Z_1 + Z_2 \tag{10.5}$$

where Z_1 is proportional to $Y-X$ and Z_2 is orthogonal to $Y-X$. The ellipsoid E^k_{XY} that had been defined by

$$|Z-X| + |Z-Y| < \frac{1}{k}|Y-X| \tag{10.6}$$

then has the "equation"

$$k^2|Z_1|^2 + \frac{k^2}{1-k^2}|Z_2|^2 < \frac{1}{4}|Y-X|^2 . \tag{10.7}$$

We apply the linear mapping

$$f(Z) = f\left(\frac{X+Y}{2} + Z_1 + Z_2\right) = \frac{X+Y}{2} + Z_1 + \frac{1}{1-k^2} Z_2 \tag{10.8}$$

which transforms the ellipsoid (10.7) into the ball

$$\left|f(Z) - \frac{X+Y}{2}\right| < \frac{1}{2k}|Y-X| ,$$

and moreover is 1-1 and (m,M)-isometric with $m=1$, $M = \frac{1}{\sqrt{1-k^2}}$, and thus of eccentricity k.

In order to prove that there is no mapping with eccentricity less than k taking E^k_{XY} into a ball we consider such a mapping taking E^k_{XY} into the ball of radius 1. The inverse mapping will be (m,M)-isometric with certain m,M. By Theorem I, p. 9, and Lemma I, p. 7, the ellipsoid E^k_{XY} contains a ball of radius m and is contained in a ball of radius M; hence its minor axis is at least $2m$, its major axis at most $2M$. Consequently $k \leq \sqrt{1 - m^2M^{-2}}$, which was to be proved.

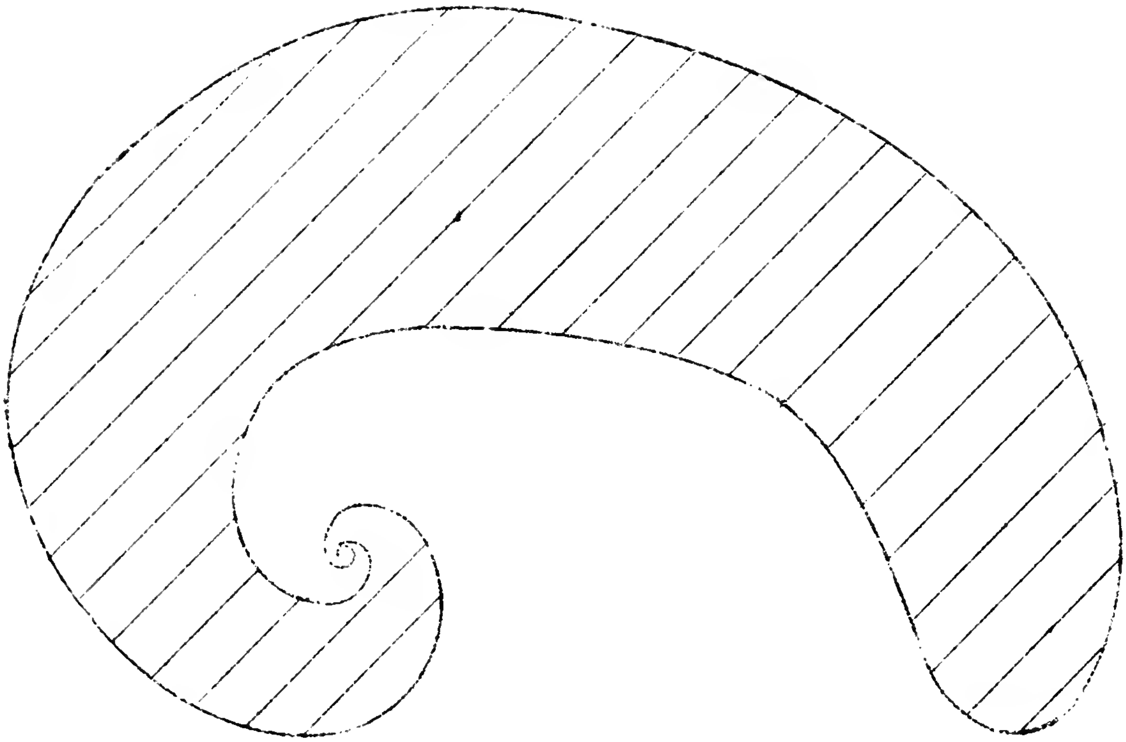


Figure 6. Plane spheroid.

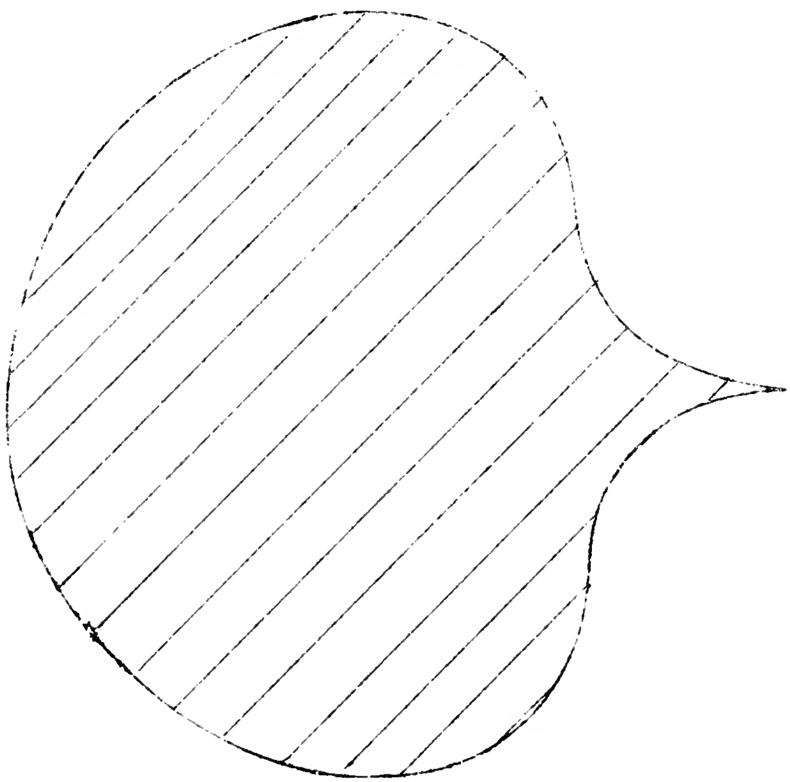


Figure 7. Plane non-spheroid.

Spheroids can be rather complicated looking sets. Figure 6 illustrates the plane set obtained from the circle with the equation $r = 2\cos\theta$ in polar coordinates by the mapping

$$(r,\theta) \to (r,\ \theta - \frac{\pi}{2}\ \frac{\log r}{\log 2})$$

This spheroid has a pointed end bounded, in the limit, by two logarithmic spirals.

An obvious necessary condition for a set R to be a spheroid is that R is <u>bounded</u>, since there has to exist a uniformly Lipschitz continuous mapping of a ball onto R. Less obvious is the fact that the boundary points of a spheroid can be at worst "conical" in the sense of the following theorem:

<u>Theorem VII</u>.

<u>Let R be a spheroid of eccentricity k</u>. <u>Let for any Y in R the distance of Y from the boundary of R be denoted by</u> $\rho(Y)$. <u>Then for every boundary point X of R</u>

$$\overline{\lim_{\substack{Y \to X \\ Y \in R}}}\ \frac{\rho(Y)}{|Y-X|} \geqq \sqrt{1-k^2} > 0 . \tag{10.9}$$

This shows that regions with a sharp point (as in Figure 7) cannot be spheroids.

Proof: Let f be an (m,M)-isometric one-one mapping of the open set R onto a ball of radius r and center z. If X is a boundary point of R we can find points $X_n \in R$ for which $\lim_{n\to\infty} X_n = X$. Put $x_n = f(X_n)$, so that $|x_n - z| < r$.

For any y with $|y-z| < r$ the inverse $Y = f^{-1}(y)$ is defined, and lies in R. Moreover, since f^{-1} is (M^{-1},m^{-1})-isometric in

the ball we have by lemma I, p. 7, that

$$\rho(Y) > \frac{1}{M}(r - |y-z|) . \tag{10.10}$$

Now

$$\rho(X_n) \leqq |X_n - X|$$

since X is a boundary point. Hence by (10.10)

$$r - |x_n - z| \leqq M|X_n - X| .$$

It follows that

$$\lim_{n\to\infty} |x_n - z| = r . \tag{10.11}$$

Introduce now the points

$$y_n = z + (1 - |X_n - X|^{1/2})(x_n - z)$$

Clearly $|y_n - z| < r$ for all sufficiently large n. Put for those n

$$Y_n = f^{-1}(y_n).$$

Then, by lemma I, p. 7

$$|Y_n - X| \leqq |Y_n - X_n| + |X_n - X| \leqq \frac{1}{m}|y_n - x_n| + |X_n - X| \tag{10.12}$$

$$= \frac{1}{m}|y_n - x_n| \left(1 + \frac{m|X_n - X|^{1/2}}{|x_n - z|}\right)$$

whereas by (10.10)

$$\rho(Y_n) \geqq \frac{1}{M}(r - |y_n - z|) \geqq \frac{1}{M}(|x_n - z| - |y_n - z|) = \frac{1}{M}|y_n - x_n| .$$

It follows that

$$\frac{\rho(Y_n)}{|Y_n-X|} \geqq \frac{m}{M}\left(1+\frac{m|X_n-X|^{1/2}}{|x_n-z|}\right)^{-1},$$

and thus, by (10.11)

$$\overline{\lim_{n\to\infty}} \frac{\rho(Y_n)}{|Y_n-X|} \geqq \frac{m}{M}.$$

Since also by (10.12)

$$\overline{\lim_{n\to\infty}} |Y_n-X| \leqq \frac{1}{m} \overline{\lim_{n\to\infty}} |y_n-x_n|$$

$$= \frac{1}{m} \overline{\lim_{n\to\infty}} |X_n-X|^{1/2}|x_n-z| = 0,$$

we have

$$\lim_{n\to\infty} Y_n = X,$$

and have proved that

$$\overline{\lim_{\substack{Y\to X\\ Y\in R}}} \frac{\rho(Y)}{|Y-X|} \geqq \frac{m}{M};$$

since $\sqrt{1-k^2}$ is the supremum of all m/M we have proved (10.9).

Theorem VIII.

Let R be a bounded open set in Hilbert-space B. Let R contain a ball from each point of which R appears star-shaped; (that is every point of R can be joined to every point of the ball by a line segment in R). Then R is a spheroid. In particular: every bounded open convex set is a spheroid.

Proof:

Assume, without restriction of generality, that R appears

star-shaped from all points of the ball $|X| < a$. Since R is bounded there will exist a concentric ball $|X| < b$ which contains all of R.

R is star-shaped from the origin and bounded. We can then describe R completely by the non-negative scalar function ϕ defined by

$$\phi(X) = \left(\sup_{\lambda X \in R} \lambda \right)^{-1} \quad \text{for} \quad X \neq 0 \tag{10.13a}$$

$$\phi(0) = 0 . \tag{10.13b}$$

This function is homogeneous of degree 1:

$$\phi(\mu X) = \mu \phi(X) \quad \text{for} \quad \mu \geqq 0 .$$

The points X of R are precisely those for which

$$\phi(X) < 1 . \tag{10.14}$$

The obvious candidate for a mapping of R onto the unit ball is the mapping given by the expression

$$x = f(X) = \frac{\phi(X)}{|X|} X \quad \text{for} \quad X \neq 0 , \qquad f(0) = 0. \tag{10.15}$$

which is linear along each ray from the origin. Using (10.14) we see immediately that f maps R one-one onto the ball $|x| < 1$. The assumption that R contains the ball of radius a about 0 and is contained in the ball of radius b about 0 yields the inequalities

$$\frac{1}{b}|X| \leqq \phi(X) \leqq \frac{1}{a}|X| \tag{10.16}$$

To make sure that f is a regular mapping we need only show that f and its inverse f^{-1} are continuous. It suffices to show that the function $\phi(X)$ is continuous. For then f(X) is continuous by (10.15), (10.16), and similarly f^{-1} continuous since

$$(10.17)\quad X = f^{-1}(x) = \frac{|x|}{\phi(x)}\,x \quad \text{for} \quad x \neq 0\,, \quad f^{-1}(0) = 0\,.$$

We shall prove that ϕ is even Lipschitz continuous.

The assumption that R is star-shaped from any point X in the ball $|X| < a$ implies that

$$(10.18)\quad \phi((1-\theta)X+\theta Y) < 1 \quad \text{for} \quad 0 \leqq \theta \leqq 1\,, \ |X| < a\,, \ \phi(Y) < 1\,.$$

(See Fig. 8) Let now Y be any point with $\phi(Y) < 1$ and let Z be arbitrary. For any θ with

$$0 < \theta < \frac{a}{|Z| + a}$$

we have

$$\left|\frac{\theta}{1-\theta}\,Z\right| < a$$

Applying (10.18) with $X = \theta Z/(1-\theta)$ we find that

$$1 > \phi(\theta Z+\theta Y) = \theta\phi(Z+Y)\ .$$

For $\theta \to a/(|Z|+a)$ we find that generally

$$\phi(Z+Y) \leqq 1 + \frac{1}{a}|Z| \quad \text{for} \quad \phi(Y) < 1\ .$$

Replacing Y and Z by λY and λZ we have then that

$$\lambda\phi(Z+Y) \leqq 1 + \frac{\lambda}{a}|Z| \quad \text{for} \quad \lambda \geqq 0\,, \ \lambda\phi(Y) < 1\ .$$

For $\lambda \to 1/\phi(Y)$ this yields the inequality

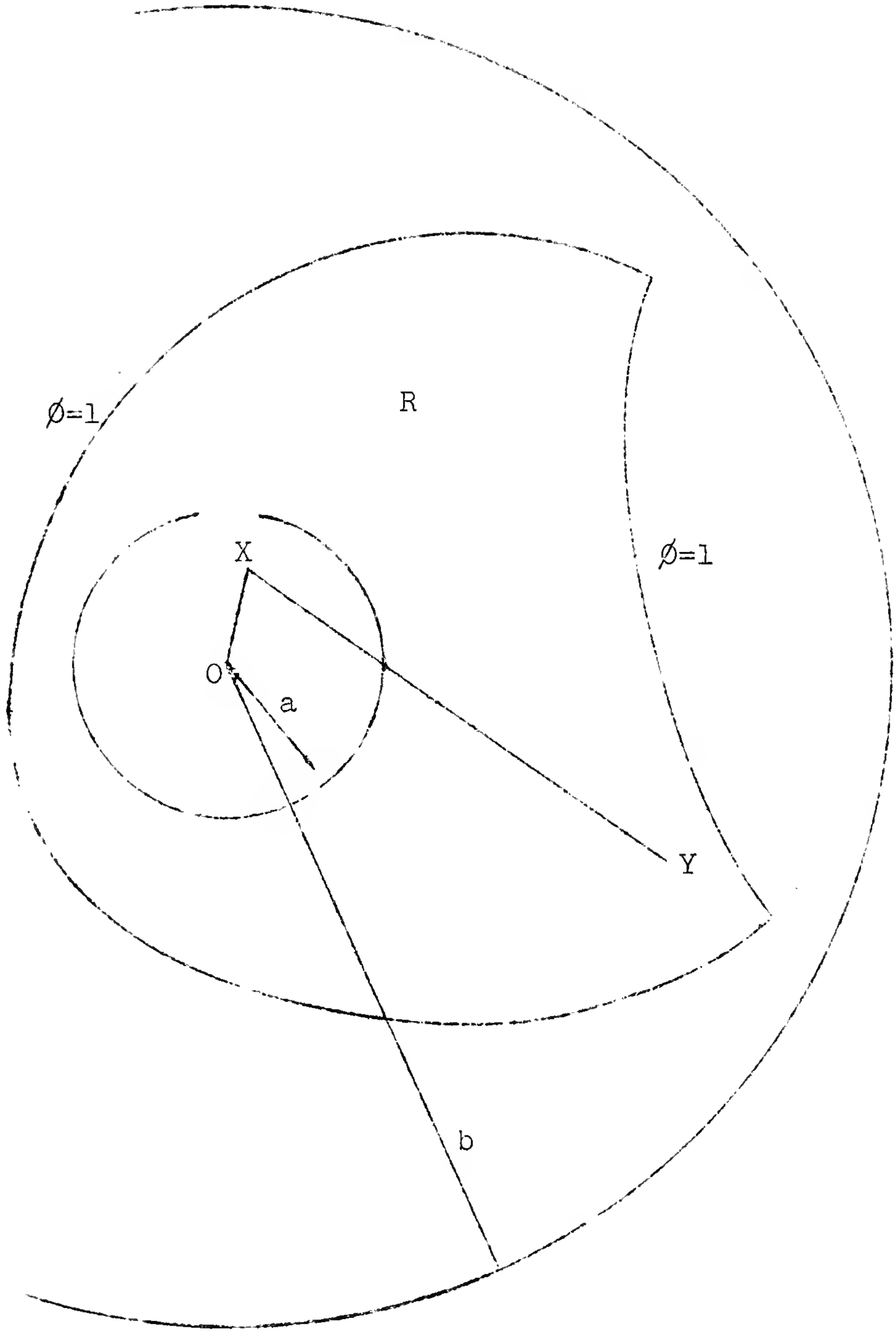


Figure 8.

$$\phi(Z+Y) \leqq \phi(Y) + \frac{1}{a}|Z|$$

provided $Y \neq 0$. But this inequality also holds for $Y = 0$ by (10.16). Putting $Z + Y = X$, and also considering the same inequality with X and Y interchanged we conclude that the function ϕ satisfies the Lipschitz condition

$$(10.19) \qquad |\phi(Y) - \phi(X)| \leqq \frac{1}{a}|Y-X| \ .$$

Since also

$$\overline{\lim_{Y \to X}} \frac{\left||Y|^{-1}Y - |X|^{-1}X\right|}{|Y-X|} = |X|^{-1}$$

we find for the function f given by (10.15), using (10.16), that

$$(10.20) \qquad M = \overline{\lim_{Y \to X}} \frac{|f(Y)-f(X)|}{|Y-X|} \leqq \frac{2}{a}$$

Similarly we have

$$\overline{\lim_{y \to x}} \frac{\left||y|y - |x|x\right|}{|y-x|} = 2|x|$$

$$\overline{\lim_{y \to x}} \frac{1}{|y-x|} \left| \frac{1}{\phi(y)} - \frac{1}{\phi(x)} \right|$$

$$= \frac{1}{\phi^2(x)} \overline{\lim_{y \to x}} \frac{|\phi(y)-\phi(x)|}{|y-x|} \leqq \frac{1}{a\phi^2(x)}$$

Hence by (10.17), (10.16)

$$\frac{1}{m} = \overline{\lim_{y \to x}} \frac{|f^{-1}(y)-f^{-1}(x)|}{|y-x|}$$

$$\leqq \frac{2|x|}{\phi(x)} + \frac{|x|^2}{a\phi^2(x)} \leqq 2b + \frac{b^2}{a} \ .$$

It follows that the mapping f is quasi-isometric with

(10.21) $$\frac{M}{m} \leqq \frac{4ab + 2b^2}{a^2} \quad ,$$

and, hence, that R is a spheroid of eccentricity

(10.22) $$k \leqq \sqrt{1 - \frac{a^4}{4(2ab + b^2)^2}} \quad .$$

11. Stiffness of sets.

Let R be an open set and X,Y two distinct points of R. For given m,M with $0 < m \leqq M < \infty$ we consider the class of all (m,M)-isometric mappings f of R and define

(11.1) $$M'(m,M,R,X,Y) = \sup_f \frac{|f(Y)-f(X)|}{|Y-X|}$$

(11.2) $$m'(m,M,R,X,Y) = \inf_f \frac{|f(Y)-f(X)|}{|Y-X|} \quad .$$

It is clear that m' and M' are homogeneous of first degree in m,M, since on replacing f(X) by $\lambda f(X)$ with a positive λ the quantities m,M are just replaced by $\lambda m, \lambda M$. Any special (m,M)-isometric mapping f of R furnishes a lower bound on M and an upper bound on m. Using the linear mappings f(X) = MX respectively f(X) = mX we see in particular that

(11.3) $$m'(m,M,R,X,Y) \leqq m \ , \quad M'(m,M,R,X,Y) \geqq M \ .$$

If R is convex we have by lemma I that M'(m,M,R,X,Y) = M . More generally the quantity M'(m,M,R,X,Y) is always finite, if R is connected. For then there are polygonal arcs connecting X and Y;

since the length of each side of such an arc is increased at most M-fold by the mapping we have

$$(11.4) \qquad M'(m,M,R,X,Y) \leqq \frac{L(X,Y,R)}{|Y-X|} M$$

where $L(R,X,Y)$ is the infinum of the lengths of all polygonal arcs connecting the points X and Y inside R. By Theorem VI, p. 27, we also have $m'(m,M,X,Y) = m$ if R contains the ellipsoid E^k_{XY} with k given by (9.1).

Since m' and M' are homogeneous in m,M the ratio M'/m' depends only on M/m and R,X,Y. We put again

$$(11.5a) \qquad \frac{M}{m} = (1+\varepsilon)^2 .$$

We can define then a quantity ε' by $M'/m' = (1+\varepsilon')^2$, that is

$$(11.5b) \qquad \varepsilon'(\varepsilon,R,X,Y) = \sqrt{\frac{M'(m,M,R,X,Y)}{m'(m,M,R,X,Y)}} - 1 .$$

We now define the <u>stiffness of the set R with respect to the points</u> X,Y by

$$(11.6) \qquad s(\varepsilon,R,X,Y) = \frac{\varepsilon}{\varepsilon'(\varepsilon,R,X,Y)}$$

Here $\varepsilon'(\varepsilon,R,X,Y)$ is a measure for the greatest relative change in distance of the points X,Y that can be obtained by quasi-isometric deformations of R without causing strains exceeding ε somewhere in R. By (11.3) we have always

$$(11.7) \qquad 0 \leqq s(\varepsilon,R,X,Y) \leqq 1 .$$

Since the restriction of an (m,M)-isometric mapping to an open subset of R is again (m,M)-isometric, we see that stiffness can

only increase with increasing R ; more precisely we have

$$(11.8) \qquad s(\varepsilon,R,X,Y) \leqq s(\varepsilon,R^*,X,Y) \quad \text{for} \quad R \subset R^* .$$

By Theorem VI, p. 27, we have $s(\varepsilon,R,X,Y) = 1$ if R contains the ellipsoid E^k_{XY} with foci X,Y and eccentricity

$$(11.9) \qquad k = \sqrt{\frac{2}{1+(1+\varepsilon)^4}} .$$

This implies in particular that a convex open set R will have stiffness 1 with respect to two given points X,Y in it for all ε that are sufficiently small. The assumption of convexity is essential. Take for example the plane referred to polar coordinates r,θ and slit along the ray $\theta = \pm\frac{\pi}{2}$. In the resulting open set R characterized by $r > 0$, $-\pi < \theta < \pi$ we consider the mapping

$$(11.10) \qquad (r,\theta) \rightarrow (r,(1+\varepsilon)^2\theta) ,$$

where ε is a positive number. Clearly for this quasi-isometric mapping of R we have $m = 1$, $M = (1+\varepsilon)^2$, and $M/m = (1+\varepsilon)^2$ in agreement with (11.5a). Consider now a point

$$X = (r,\theta) \quad \text{with} \quad \frac{\pi}{2} < \theta < \pi$$

and take for Y the symmetric point

$$Y = (r,-\theta) .$$

Then for ε sufficiently small

$$|Y-X| = 2r \sin\theta , \quad |f(Y)-f(X)| = 2r \sin (1+\varepsilon)^2\theta$$

Hence

$$m'(m,M,R,X,Y) \leqq \frac{\sin (1+\varepsilon)^2\theta}{\sin\theta} .$$

Since also $M'(m,M,R,X,Y) \geqq M$, we have

$$(11.11) \qquad (1+\varepsilon')^2 = \frac{M'(m,M,R,X,Y)}{m'(m,M,R,X,Y)} \geqq \frac{(1+\varepsilon)^2 \sin\theta}{\sin(1+\varepsilon)^2\theta}$$

$$= 1+2(1-\theta\cot\theta)\varepsilon + O(\varepsilon^2)$$

Hence

$$\varepsilon' \geqq (1-\theta\cot\theta)\,\varepsilon + O(\varepsilon^2)$$

and consequently

$$(11.12) \qquad \overline{\lim_{\varepsilon\to 0}}\; s(\varepsilon,R,X,Y) \leqq \frac{1}{1-\theta\cot\theta} < 1$$

since $\cot\theta < 0$ by assumption on the point X. Since stiffness increases with set R the inequality (11.12) still holds for the same points X,Y and any open set R that contains X,Y and is contained in the slit plane used previously. (See Fig. 8.)

We define now a stiffness depending only on the region R and on ε by

$$(11.13) \qquad s(\varepsilon,R) = \inf_{X,Y\in R} s(\varepsilon,R,X,Y)\ .$$

If R is the whole space we have $s(\varepsilon,R) = 1$ for all $\varepsilon \geqq 0$ by Corollary I, p. 11. But for general regions this cannot be expected. To see this we take again the mapping (11.10), but this time applied to the open half-plane $|\theta| < \pi/2$. We take again two symmetric points $X = (r,\theta)$ and $Y = (r,-\theta)$, where now θ is some value with $0 < \theta < \pi/2$. We find as before the inequality (11.11) for $\varepsilon'(\varepsilon,R,X,Y)$. Since here θ, depending on the choice of X, is any number between 0 and $\pi/2$, we have for $\theta \to \pi/2$

$$\inf_{X,Y\in R} (1+\varepsilon')^2 \geqq \frac{(1+\varepsilon)^2}{\sin\frac{\pi}{2}(1+\varepsilon)^2}$$

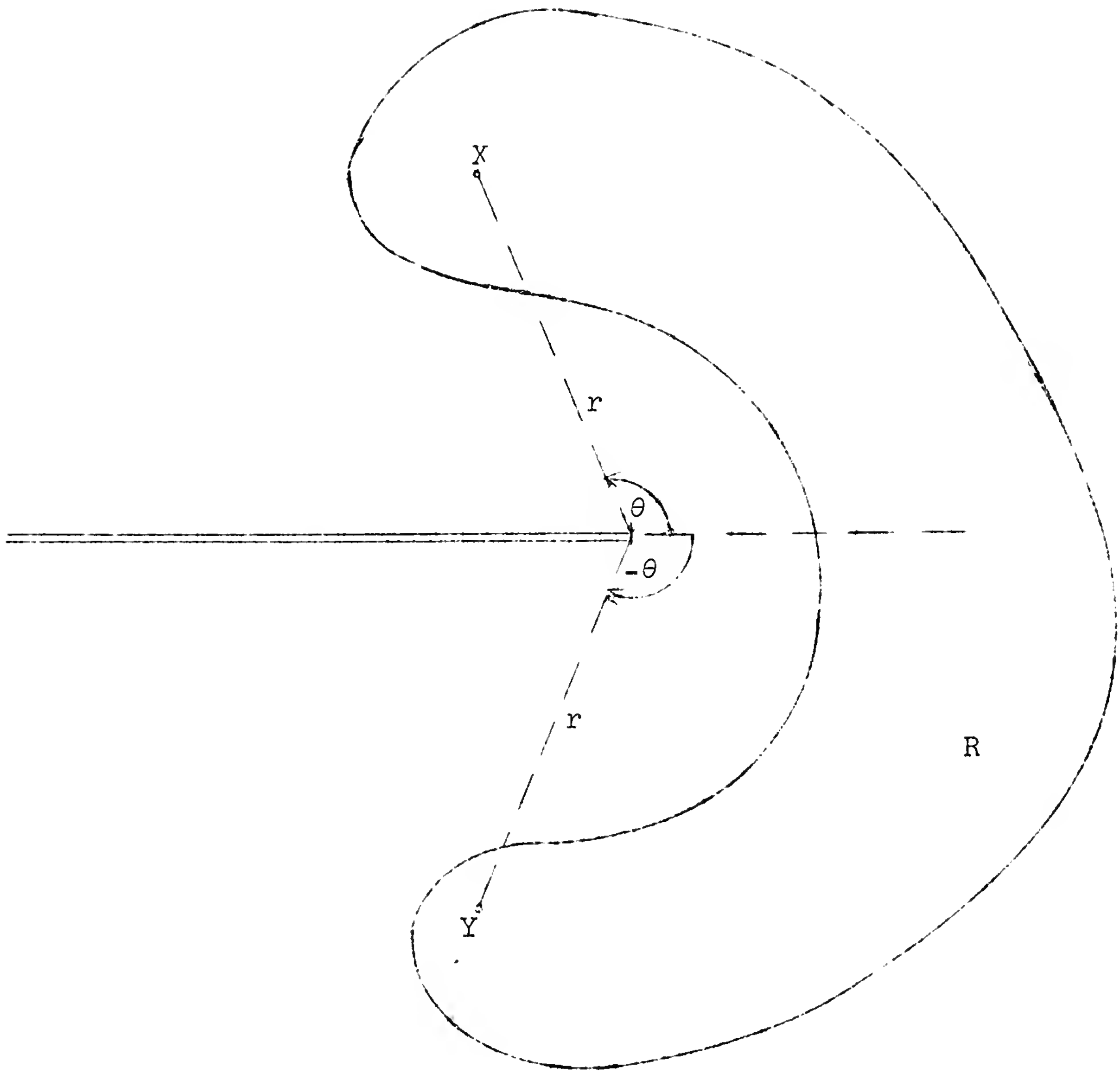


Figure 9.

and hence

$$(11.14) \qquad s(\varepsilon,R) \leqq \frac{\varepsilon}{(1+\varepsilon)\sin^{-1/2}[\frac{\pi}{2}(1+\varepsilon)^2]-1}$$

$$= 1 - \frac{\pi^2}{4}\varepsilon + O(\varepsilon^2)$$

In particular we find that the stiffness $s(\varepsilon,R)$ is zero for $\varepsilon = \sqrt{2}-1$; for strains of this size R has no stiffness, i.e. we can find points in R with arbitrarily large relative change in distance.

The estimate (11.14) for the stiffness of the region R has been established for the case where R is a 2-dimensional open half-plane. The result can be extended immediately to the more general case where R is a half-plane in Hilbert-space of dimension > 1. Here a half-space R is described by the set of X satisfying an inequality of the form $(X-X_0)\cdot Z > 0$, where Z is a fixed non-vanishing element of length 1 of the space. For the proof we only have to provide a mapping analogous to (1.10). For this purpose we select a unit-vector T orthogonal to Z. In the two-dimensional plane

$$(11.15) \qquad X = X_0 + \lambda Z + \mu T \qquad (\lambda,\mu \text{ arbitrary real numbers})$$

we introduce polar coordinates r,θ by

$$r = \sqrt{\lambda^2+\mu^2}\,, \quad \cos\theta = \frac{\lambda}{r}\,, \quad \sin\theta = \frac{u}{r}\,.$$

We then apply the transformation taking r,θ respectively into r and $(1+\varepsilon)^2\theta$, and which keeps the component of $X-X_0$ orthogonal to Z and T fixed. The resulting transformation is (m,M)-isometric

in the half-space, with $m = 1$, $M = (1+\varepsilon)^2$, and all previous conclusions apply. As a matter of fact the estimate (11.14) for the stiffness of R holds then for much more general regions R, namely for all R which are contained in a half-space and contain a ball touching the boundary of the half-space. That is, we have a point X_o, a unit vector Z and a radius ρ such that $(X: |X-X_o-aZ| < \rho) \subset R \subset (X: (X-X_o)\cdot Z > 0)$. Introducing polar coordinates as before we see that R will contain pairs of points X,Y in the two-dimensional plane (11.15) which are symmetric to the axis $\theta = 0$ and have $|\theta|$ arbitrarily close to $\pi/2$. In particular the estimate (11.14) applies to the case where R itself is a ball.

Even a ball then will have no stiffness for strains as large as $\varepsilon = \sqrt{2} - 1$. A lower bound for the stiffness of a ball is supplied by Corollary VII, p.32, according to which

$$m' \geqq \mu \ , \qquad M' = M$$

with μ given by (9.12). This gives the estimate

$$(1+\varepsilon')^2 \leqq \frac{(1+\varepsilon)^2\left(1+\sqrt{\frac{1}{2}((1+\varepsilon)^4-1)}\right)}{1-(1+\varepsilon)^2\sqrt{\frac{1}{2}((1+\varepsilon)^4-1)}}$$

This results for very small ε in a very poor estimate for the stiffness:

$$s(\varepsilon,R) \geqq \sqrt{\frac{\varepsilon}{2}} + O(\varepsilon)$$

which will be improved later.

12. Stiffness of pins.

Lemma VI:

Let f be an (m,M)-isometric mapping of a ball

$$|X-X_o| < r \tag{12.1}$$

where

$$\frac{M}{m} = (1+\varepsilon)^2 \tag{12.2}$$

Let X_1, X_2 be two points in the smaller concentric ball $|X-X_o| < (1+\varepsilon)^{-2}r$ lying on opposite radii; that is for the quantities

$$A_i = |X_i - X_o| \;, \qquad i = 1,2 \tag{12.3}$$

we have

$$X_o = \frac{A_1X_2 + A_2X_1}{A_1+A_2} \;; \quad 0 < A_i < (1+\varepsilon)^{-2}r \quad \text{for } i=1,2. \tag{12.4}$$

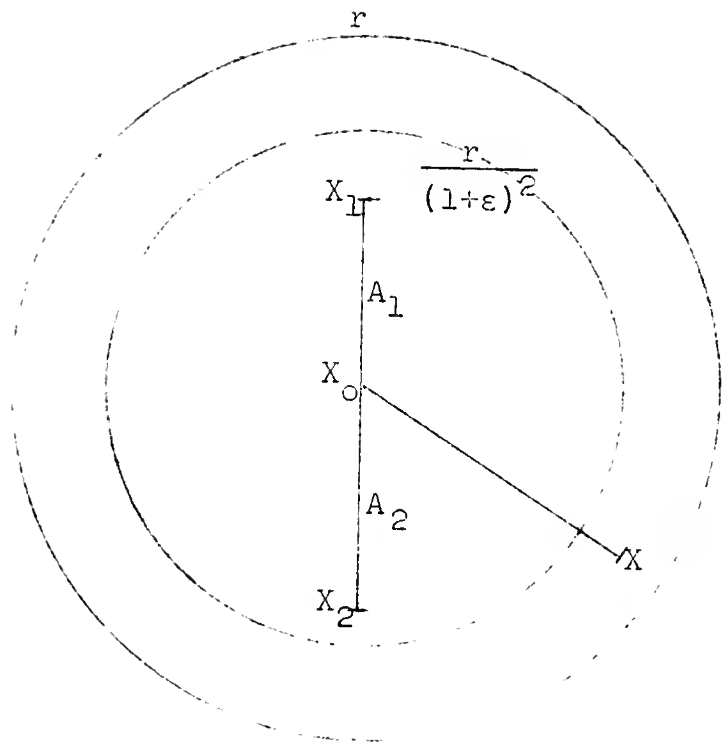
Put

$$x_i = f(X_i) \qquad \text{for} \quad i = 0,1,2 \tag{12.5}$$

and let ϕ be the angle of the triangle with vertices x_o, x_1, x_2 at the vertex x_o defined to have a value with $0 \leqq \phi \leqq \pi$. Then

$$\phi \geqq \pi - 2\pi\varepsilon \left(\sqrt{A_1/A_2} + \sqrt{A_2/A_1}\right). \tag{12.6}$$

(See Fig. 10. The points X_1, X_o, X_2 are collinear. The angle $\pi - \phi$ measures the deviation from collinearity of the image points x_1, x_o, x_2. Formula (12.6) shows that for sufficiently small ε the points stay approximately collinear.)


r
$\frac{r}{(1+\varepsilon)^2}$
X_1
A_1
X_o
A_2
X_2
X

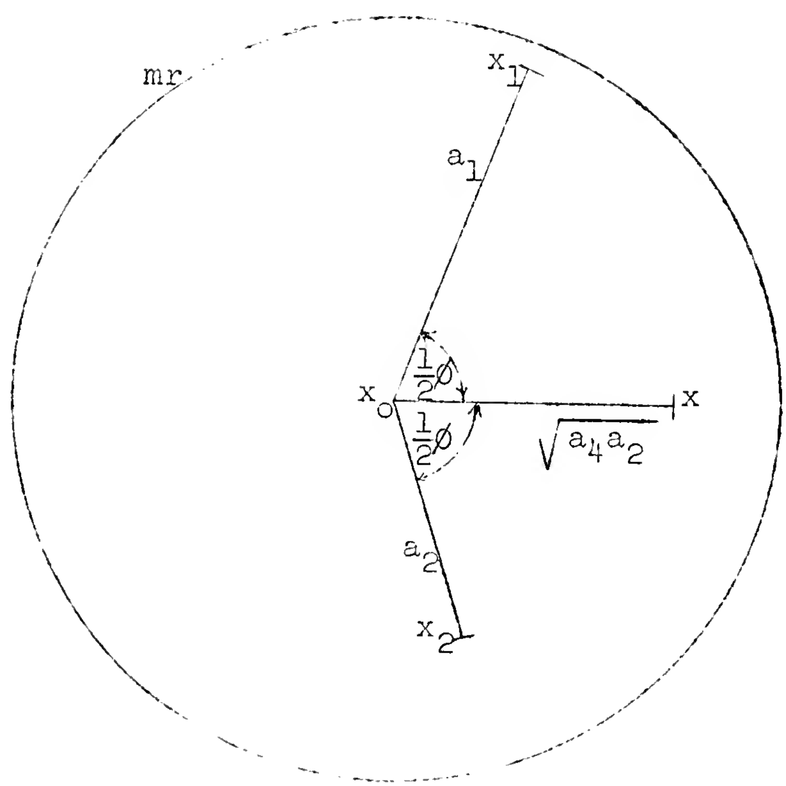

mr
x_1
a_1
$\frac{1}{2}\phi$
x_o
$\frac{1}{2}\phi$
x
$\sqrt{a_4 a_2}$
a_2
x_2

Figure 10.

Proof:

Put

(12.7) $$a_i = |x_i - x_o| \quad \text{for} \quad i = 1, 2.$$

From the proof of Theorem II, p. 10, it is apparent that we have an inverse mapping $f^{-1}(x) = f_{X_o}^{-1}(x)$ defined throughout the ball

(12.8) $$|x - x_o| < mr$$

and that

(12.9) $$X_i = f^{-1}(x_i) \quad \text{for} \quad i = 0, 1, 2.$$

Clearly the 3 points x_o, x_1, x_2 are distinct, since X_o, X_1, X_2 are distinct. If $\phi = \pi$ nothing is to be proved. If $0 \leqq \phi < \pi$ there is defined a unique ray from x_o in the same plane as x_o, x_1, x_2 which bisects the angle ϕ. Let x be that point of the ray for which

$$|x - x_o| = \sqrt{a_1 a_2} \ .$$

Then x satisfies (12.8) since by Theorem II, p. 10

$$mA_i \leqq a_i \leqq MA_i$$

and thus by (12.4)

$$|x - x_o| \leqq M \sqrt{A_1 A_2} \leqq M(1+\varepsilon)^{-2} r = mr$$

Hence $f^{-1}(x) = X$ is defined and satisfies (12.1).

At least one of the supplementary angles X_1, X_o, X and X_2, X_o, X is not acute. Let it be the first one. Then

$$|X_1 - X_o|^2 + |X - X_o|^2 \leqq |X_1 - X|^2$$

Consequently, since f^{-1} is (M^{-1},m^{-1})-isometric,

$$|x_1-x_o|^2+|x-x_o|^2 \leqq M^2(|X_1-X_o|^2+|X-X_o|^2)$$

$$\leqq M^2|X_1-X|^2 = M^2|f^{-1}(x_1)-f^{-1}(x)|^2$$

$$\leqq \frac{M^2}{m^2}|x_1-x|^2 = (1+\varepsilon)^4|x_1-x|^2$$

It follows that

$$\cos\frac{1}{2}\phi = \frac{|x_1-x_o|^2+|x-x_o|^2-|x-x_1|^2}{2|x_1-x_o|\ |x-x_o|}$$

$$\leqq \frac{(1+\varepsilon)^4-1}{(1+\varepsilon)^4}\ \frac{|x_1-x_o|^2+|x-x_o|^2}{2|x_1-x_o|\ |x-x_o|}$$

$$= \frac{(1+\varepsilon)^4-1}{2(1+\varepsilon)^4}\left(\sqrt{\frac{a_1}{a_2}}+\sqrt{\frac{a_2}{a_1}}\right)$$

$$\leqq \frac{(1+\varepsilon)^4-1}{2(1+\varepsilon)^3}\left(\sqrt{\frac{A_1}{A_2}}+\sqrt{\frac{A_2}{A_1}}\right)$$

$$\leqq 2\varepsilon\left(\sqrt{\frac{A_1}{A_2}}+\sqrt{\frac{A_2}{A_1}}\right)$$

Then

$$\pi-\phi = 2\left(\frac{\pi}{2}-\frac{\phi}{2}\right) \leqq \pi\sin\left(\frac{\pi}{2}-\frac{\phi}{2}\right) = \pi\cos\frac{1}{2}\phi ,$$

which implies (12.6).

<u>Theorem IX</u>.

<u>Let R be a convex open set containing the ball</u> $|X-X_o| < \alpha$ <u>and contained in the concentric ball</u> $|X-X_o| < \beta$. <u>Then for any X in R we have the estimate</u>

$$s(\varepsilon, R, X_o, X) \geqq 1 - c\,\frac{\beta^2}{\alpha^2}\,\varepsilon \tag{12.10}$$

<u>for the stiffness of R with respect to</u> X_o, X <u>where c is a universal constant</u>.

The theorem applies in particular to "pins", that is regions consisting of the convex hull of a ball $|X-X_o| < \alpha$ and of a point Y outside the ball where $|Y-X_o| = \beta$. For strains ε small compared to the square of the thickness-length ratio α/β the stiffness of the pin with respect to the "center" X_o and "point" Y is near 1. For strains of the order α^2/β^2 the stiffness is essentially diminished. This can be seen from the example of the "pin" R in the complex Z-plane consisting of the convex hull of the disk $|Z| < a$ and of the point $Z = i\beta$. We subject it to the conformal mapping

$$z = \frac{e^{Z\alpha^{-1}\log(1+\varepsilon)}}{\alpha^{-1}\log(1+\varepsilon)} = f(Z)$$

which is (m,M)-isometric in the pin with

$$m = \frac{1}{1+\varepsilon}\ , \qquad M = 1+\varepsilon\ .$$

Let X_o be the origin and Y the point corresponding to the complex number $i\beta$. Then $M'(m,M,R,X_o,Y) = M = 1+\varepsilon$ because of the convexity of R, and

$$\begin{aligned} m'(m,M,R,X_o,Y) &\leqq \frac{|f(i\beta)-f(0)|}{|i\beta|} \\ &= \frac{\sin\left[\frac{\beta}{2\alpha}\log(1+\varepsilon)\right]}{\frac{\beta}{2\alpha}\log(1+\varepsilon)}\ . \end{aligned}$$

Put

$$\varepsilon = 4\mu^2 \quad \frac{\alpha^2}{\beta^2}$$

and assume that $\varepsilon << a/b << 1$.

$$m' \leqq \frac{\sin(\mu^{1/2}\log(1+\varepsilon))}{\mu\varepsilon^{-1/2}\log(1+\varepsilon)} \approx 1 - \frac{1}{6}\mu^2\varepsilon + \ldots$$

$$(1+\varepsilon')^2 = \frac{M'}{m'} \approx 1 + (1+\frac{1}{6}\mu^2)\varepsilon + \ldots$$

$$s(\varepsilon, R, X_o, Y) = \frac{\varepsilon}{\varepsilon'} \leqq \frac{2}{1+\frac{1}{6}\mu^2+\ldots} \approx \frac{2}{1+\varepsilon\beta^2/24\alpha^2}$$

For ε large compared to α^2/β^2 the stiffness s will be small.

Proof of Theorem IX. (See Fig. 11.)

Let X be a point of R. Then

(12.11) $$|X-X_o| < \beta .$$

Let f be an (m,M)-isometric mapping of R. We normalize the mapping in such a way that

(12.12) $$m = \frac{1}{1+\varepsilon} , \quad M = 1+\varepsilon .$$

Put

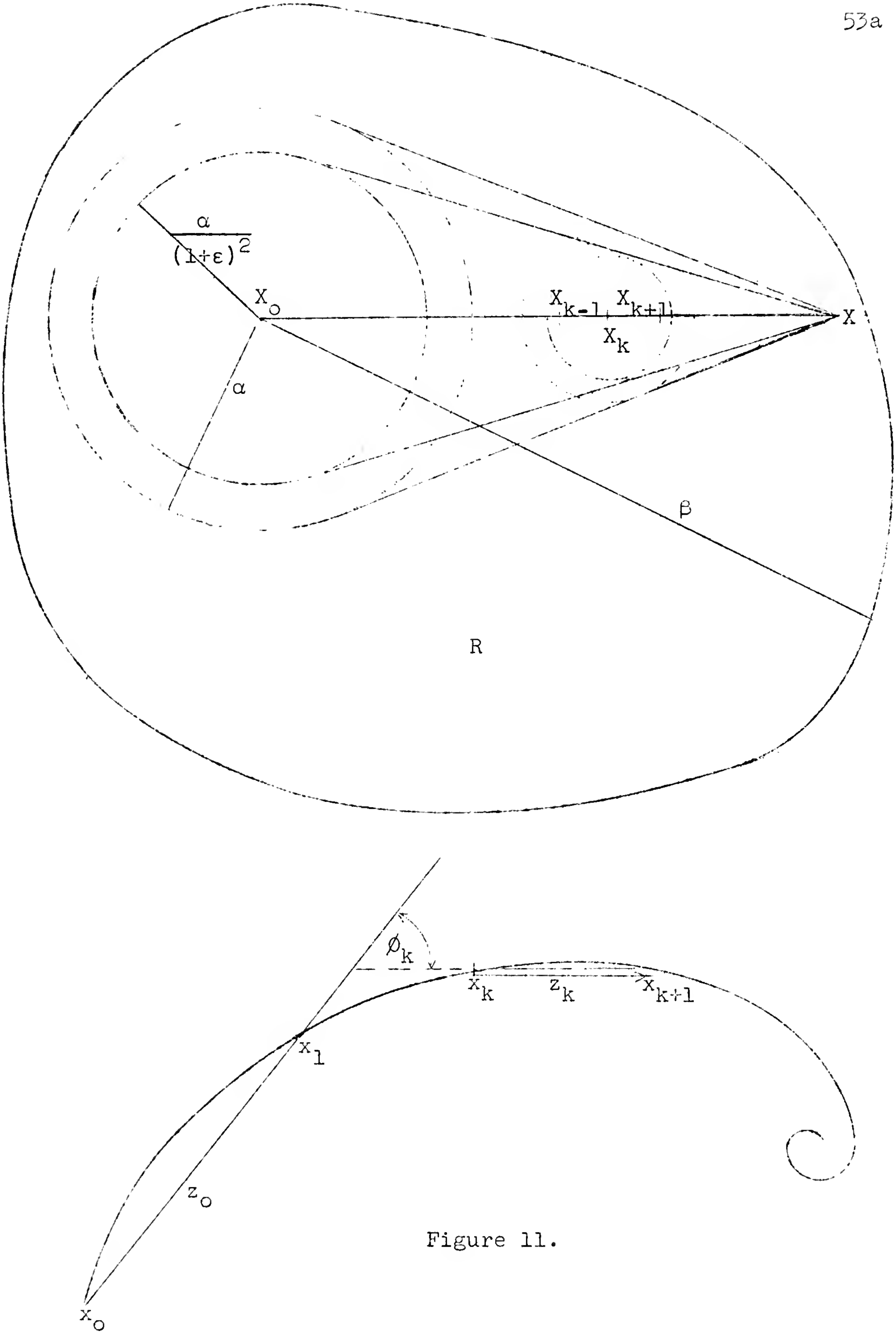
(12.13) $$q = \frac{(1+\varepsilon)^2\beta}{(1+\varepsilon)^2\beta + \alpha} .$$

Then

(12.14) $$\frac{1}{2} \leqq q < 1$$

since $0 < \alpha \leqq \beta$. We introduce the sequence of points

(12.15) $$X_k = q^k X_o + (1-q^k)X \quad \text{for} \quad k=0,1,2,3,\ldots$$



$\frac{\alpha}{(1+\varepsilon)^2}$
X_o
X_{k-1}
X_{k+1}
X_k
X
α
β
R
ϕ_k
x_k
z_k
x_{k+1}
x_1
z_o
x_o

Figure 11.

Let also

(12.16) $$Z_k = X_{k+1} - X_k \ , \quad A_k = |Z_k|$$

(12.17) $$x_k = f(X_k) \ , \quad z_k = x_{k+1} - x_k \ , \quad a_k = |z_k| \ .$$

Since R is convex and contains the point X as well as the ball of radius α about X_o we see from (12.15) that R contains the ball of radius $q^k\alpha$ about X_k as well. Moreover

(12.18a) $$A_{k-1} = |X_k - X_{k-1}| = (q^{k-1} - q^k)|X - X_o| < (\frac{1}{q} - 1)\, q^k\beta$$
$$= (1+\varepsilon)^{-2} q^k \alpha$$

(12.18b) $$A_k = |X_k - X_{k+1}| = q|X_k - X_{k-1}| < (1+\varepsilon)^{-2} q^k \alpha \ .$$

It follows then from lemma VI, p. 49, that the smallest non-negative angle between the vectors Z_k and Z_{k-1} does not exceed the value

$$\psi = 2\pi\varepsilon(q^{1/2} + q^{-1/2}) \ .$$

Then the smallest non-negative angle ϕ_k between Z_o and Z_k does not exceed the value $k\psi$, as follows from the triangle inequaltiy on spheres.

Now $X = \lim_{k\to\infty} X_k$; hence

$$f(X) - f(X_o) = \lim_{k\to\infty} (f(X_k) - f(X_o))$$

$$= \lim_{k\to\infty} (x_k - x_o) = \sum_{k=0}^{\infty} (x_{k+1} - x_k) = \sum_{k=0}^{\infty} z_k$$

Consequently

$$|f(X)-f(X_o)| \geqq \frac{(f(X)-f(X_o))\cdot z_o}{|z_o|} = \sum_{k=0}^{\infty} \frac{z_k\cdot z_o}{|z_o|}$$

$$= \sum_{k=0}^{\infty} |z_k| \cos \phi_k \geqq \sum_{k=0}^{\infty} |z_k|(1-\tfrac{1}{2}k^2\psi^2)$$

$$= \sum_{k=0}^{\infty} a_k - \tfrac{1}{2}\psi^2 \sum_{k=0}^{\infty} k^2 a_k$$

By Theorem II., p. 10, we have, using (12.18b)

$$\frac{1}{1+\varepsilon} A_k \leqq a_k \leqq (1+\varepsilon)A_k \quad .$$

Hence

$$|f(X)-f(X_o)| \geqq \frac{1}{1+\varepsilon} \sum_{k=0}^{\infty} A_k - \tfrac{1}{2}\psi^2 (1+\varepsilon) \sum_{k=0}^{\infty} k^2 A_k$$

$$= \frac{1}{1+\varepsilon} |X-X_o| - \tfrac{1}{2}\psi^2 (1+\varepsilon)(1-q)|X-X_o| \sum_{k=0}^{\infty} k^2 q^k$$

$$= \frac{1}{1+\varepsilon}\left(1 - 2\pi^2\varepsilon^2(1+\varepsilon)^2 \frac{(1+q)^3}{(1-q)^2}\right)|X-X_o|$$

Here by (12.14), (12.13)

$$1+q < \frac{3}{2}, \quad 1-q = \frac{\alpha}{(1+\varepsilon)^2\beta+\alpha} \geqq \frac{\alpha}{(1+\varepsilon)^2\beta}$$

Assume also for the moment that $\varepsilon \leqq 1$. Then

$$(1+\varepsilon)^2 \frac{(1+q)^3}{(1-q)^2} \leqq 2^9 \frac{\beta^2}{\alpha^2} \quad .$$

It follows that

$$(12.19) \qquad m'(\tfrac{1}{1+\varepsilon}, 1+\varepsilon, R, X_o, X) = \inf_f \frac{|f(X)-f(X_o)|}{|X-X_o|}$$

$$\geqq \frac{1}{1+\varepsilon}\left(1-2^{10}\pi^2\varepsilon^2 \frac{\beta^2}{\alpha^2}\right) .$$

Assume momentarily that

$$(12.20) \qquad \varepsilon < \frac{\alpha}{2^6\pi\beta} ;$$

then by (12.19)

$$m' \geqq \frac{1}{1+\varepsilon} \frac{1}{(1+2^{10}\pi^2\varepsilon^2\beta^2\alpha^{-2})^2}$$

Since also, because of the convexity of R,

$$M'(\tfrac{1}{1+\varepsilon}, 1+\varepsilon, R, X_o, X) = M = 1+\varepsilon$$

we have

$$\varepsilon'(\varepsilon, R, X_o, X) = \sqrt{\frac{M'}{m'}} - 1 \leqq (1+\varepsilon)(1+2^{10}\pi^2\varepsilon^2\beta^2\alpha^{-2}) - 1$$

$$= \varepsilon(1+2^{10}\pi^2\beta^2\alpha^{-2}(\varepsilon+\varepsilon^2)) \leqq \varepsilon(1+2^{11}\pi^2\beta^2\alpha^{-2}\varepsilon)$$

Thus, finally, under the assumption (12.20)

$$s(\varepsilon, R, X_o, X) = \frac{\varepsilon}{\varepsilon'(\varepsilon, R, X_o, X)}$$

$$\geqq \frac{1}{1+2^{11}\pi^2\beta^2\alpha^{-2}\varepsilon} \geqq 1-2^{11}\pi^2\beta^2\alpha^{-2}\varepsilon .$$

The same inequality holds trivially for $\varepsilon > \alpha/2^6\pi\beta$. Hence (12.10) is proved with $c = 2^{11}\pi^2$.

13. Stiffness of convex regions.

Theorem X.

Let R be an open convex set containing the ball $|X-X_o| < \alpha$ *and contained in the concentric ball* $|X-X_o| < \beta$. *Let f be a (m,M)-isometric mapping. Then for any points* X_1, X_2 *in* R

$$(13.1) \qquad m\left(1-3\varepsilon-C\,\frac{\beta^2}{\alpha^2}\,\varepsilon^2\right) \leqq \frac{|f(X_2)-f(X_1)|}{|X_2-X_1|} \leqq M \ ,$$

where, as before, ε *is defined by* $(1+\varepsilon)^2 = M/m$, *and where* C *is a universal constant*

Proof: Of course only the left hand part of (13.1) needs proving.

It is sufficient to prove (13.1) for the case where

$$(13.2) \qquad |X_2-X_1| \geqq \frac{1}{3}\beta \ .$$

Let indeed

$$(13.2) \qquad |X_2-X_1| < \frac{1}{3}\beta \ .$$

Introduce the point (see Fig. 12)

$$(13.3) \qquad X_o' = (1-\theta)X_o + \theta X_1 \quad ,$$

where

$$(13.4) \qquad \theta = 1 - \frac{1}{\beta}|X_2-X_1| \ .$$

lies between $\frac{2}{3}$ and 1. The ball

$$(13.5) \qquad |X-X_o'| < (1-\theta)\alpha = \alpha'$$

lies in R, since R is convex and contains the ball $|X-X_o| < \alpha$

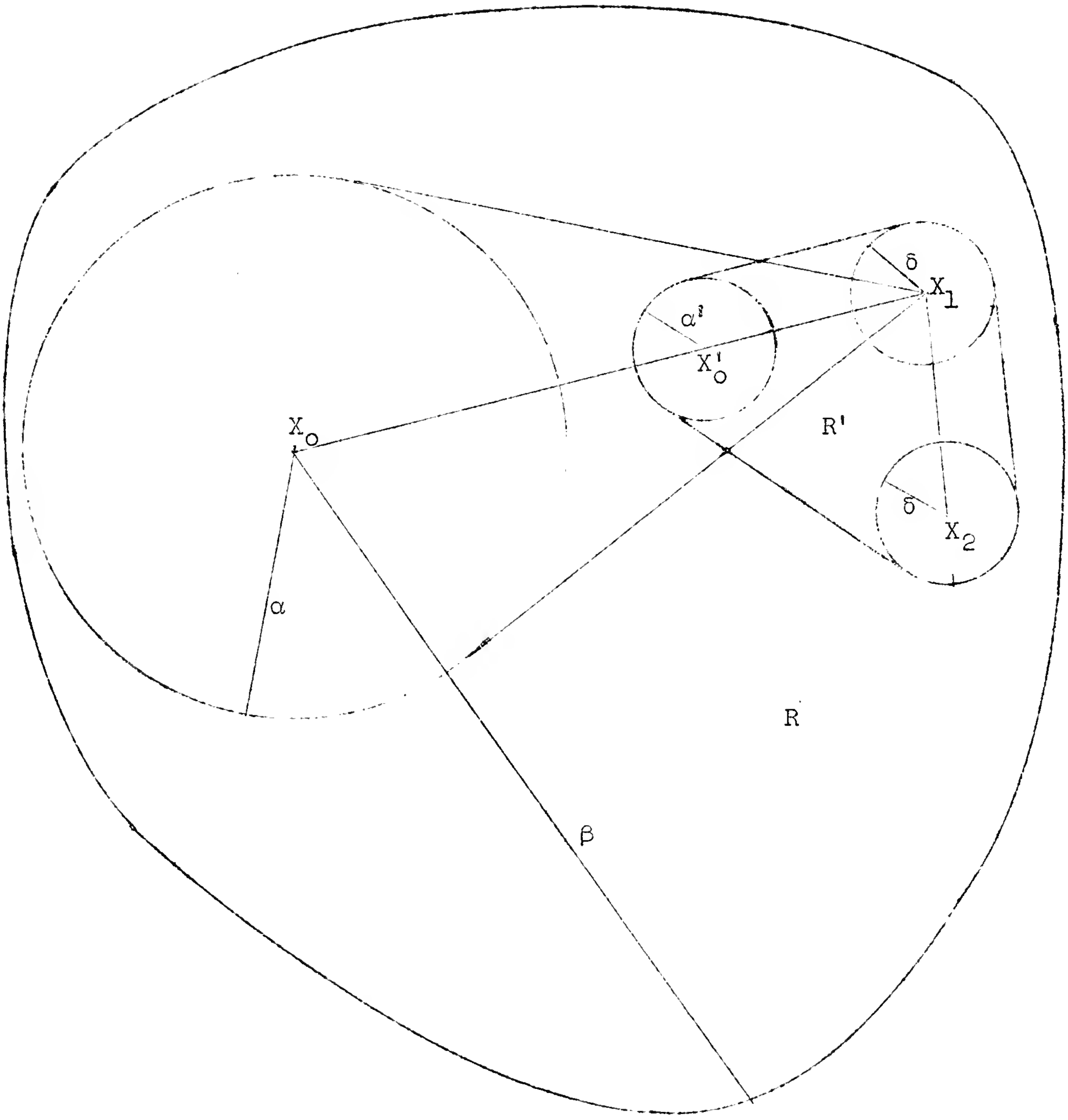


Figure 12.

and the point X_1. There also exists a positive $\delta < |X_2-X_1|$ such that the balls

$$(13.6) \qquad |X-X_1| < \delta \quad \text{and} \quad |X-X_2| < \delta$$

lie in R. Let R' be the convex hull of the three balls (13.5) and (13.6) (defined as the set of points

$$\lambda_1 Y_1 + \lambda_2 Y_2 + \lambda_3 Y_3$$

where Y_1, Y_2, Y_3 are respectively in the first, second and third ball and the λ_i are non-negative numbers of sum 1.) Then R' is a convex set containing the points X_1, X_2. Moreover, R' contains the ball (13.5) and is contained in the concentric ball $|X-X_o'| < \beta'$, where

$$\beta' = \operatorname{Max}(\alpha', |X_1-X_o'| + \delta, |X_2-X_o'| + \delta) .$$

Here

$$\alpha' = (1-\theta)\alpha = \frac{\alpha}{\beta}|X_2-X_1| \leqq |X_2-X_1|$$

$$|X_1-X_o'| = (1-\theta)|X_1-X_o| = \frac{1}{\beta}|X_1-X_o||X_2-X_1| \leqq |X_2-X_1|$$

$$|X_2-X_o'| < |X_1-X_o'| + |X_2-X_1| < 2|X_2-X_1| .$$

Hence

$$(13.6) \qquad \beta' \leqq 2|X_2-X_1| + \delta \leqq 3|X_2-X_1| ,$$

and

$$\frac{\beta'}{\alpha'} \leq \frac{3|X_2-X_1|}{(1-\theta)\alpha} = 3\frac{\beta}{\alpha} .$$

If now (13.1) had been established under the assumption (13.2)

we could apply it to the mapping f in the region R' and the points X_1, X_2 because of (13.6). It would follow that

$$\frac{|f(X_2)-f(X_1)|}{|X_2-X_1|} \geqq m(1-3\varepsilon-C\,\frac{\beta'^2}{\alpha'^2}\,\varepsilon)$$

$$\geqq m(1-3\varepsilon-9C\,\frac{\beta^2}{\alpha^2}\,\varepsilon)$$

which is the inequality to be proved, only with C replaced by another universal constant 9C.

Let us assume then that (13.2) is satisfied. Theorem IX already permits us to estimate $|f(Y)-f(X)|$ if at least one of the points is well inside the region R. We shall estimate $|f(X_2)-f(X_1)|$ by proper use of a "baseline" with endpoints Y_1, Y_2 some distance inside R from which we can survey X_1 and X_2. (See Fig. 13.)

We define the quantity μ by

(13.7) $$\mu = \frac{9}{10}$$

and introduce the auxiliary points

(13.8) $$Y_i = X_o + \mu(X_i - X_o) \quad \text{for} \quad i=1,2 \; .$$

Put

$$x_i = f(X_i) \quad \text{and} \quad y_i = f(Y_i) \quad \text{for} \quad i=1,2 \; .$$

Then

$$|f(X_2)-f(X_1)| = |x_2-x_1| \geqq \frac{(x_2-x_1)\cdot(y_2-y_1)}{|y_2-y_1|}$$

$$= \frac{|x_2-y_1|^2+|x_1-y_2|^2-|x_1-y_1|^2-|x_2-y_2|^2}{2|y_2-y_1|}$$

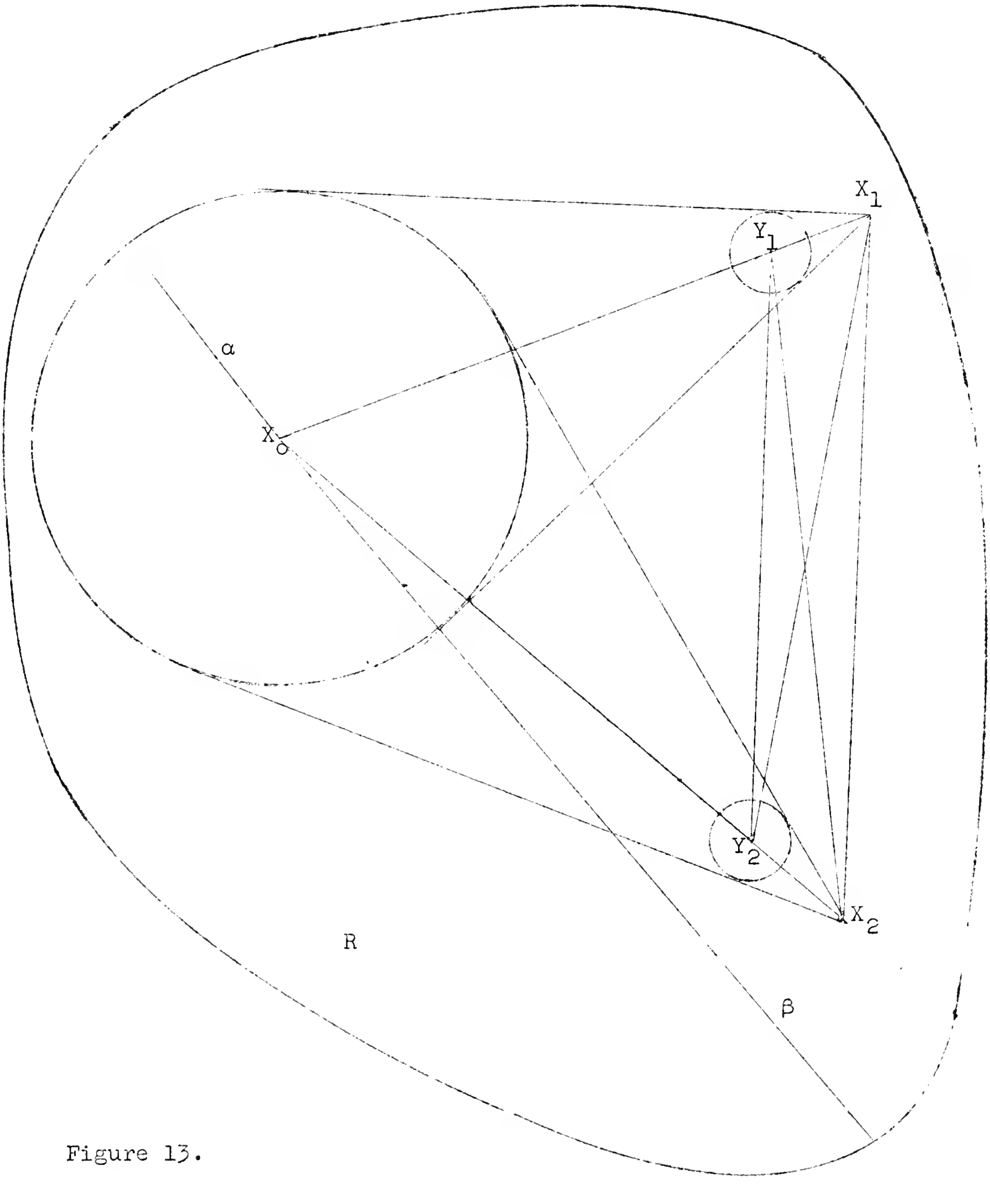


Figure 13.

Here, because of the convexity of R,

$$|x_1-y_1|^2+|x_2-y_2|^2 \leqq M^2(|X_1-Y_1|^2+|X_2-Y_2|^2) \tag{13.10}$$

$$|y_2-y_1| \leqq M|Y_2-Y_1|$$

Since R is convex and contains the ball $|X-X_o| < \alpha$ and the points X_1, X_2, we have from (13.8) that R also contains the balls

$$|X-Y_1| < (1-\mu)\alpha \quad \text{and} \quad |X-Y_2| < (1-\mu)\alpha .$$

Moreover

$$\frac{|X_i-Y_k|}{(1-\mu)\alpha} \leqq \frac{|X_i-Y_i|+|Y_i-Y_k|}{(1-\mu)\alpha}$$

$$= \frac{(1-\mu)|X_i-X_o|+\mu|X_i-X_k|}{(1-\mu)\alpha} \leqq \frac{\beta}{\alpha} + \frac{\mu}{1-\mu} \quad \frac{|X_i-X_k|}{\alpha}$$

$$\leqq \frac{\beta}{\alpha} + \frac{9}{\alpha}|X_1-X_2| \leqq \frac{\beta}{\alpha} + 18\,\frac{\beta}{\alpha}$$

$$\leqq (1 \quad \frac{2\mu}{1-\mu})\ \frac{\beta}{\alpha} = 19\,\frac{\beta}{\alpha}$$

It follows then from (12.19) that

$$|x_i-y_k| = |f(X_i)-f(Y_k)| \geqq m(1-\sigma^2\varepsilon^2\,\frac{\beta^2}{\alpha^2})|X_i-Y_k|$$

where

$$\sigma = 2^5(19)\pi . \tag{13.12}$$

Assume momentarily that

$$\sigma\varepsilon\,\frac{\beta}{\alpha} \leqq \frac{1}{2} . \tag{13.13}$$

Hence

$$|x_2-y_1|^2+|x_1-y_2|^2$$

$$\geqq m^2(1-\sigma^2\varepsilon^2\frac{\beta^2}{\alpha^2})^2(|X_2-Y_1|^2+|X_1-Y_2|^2)$$

$$= m^2(1-\sigma^2\varepsilon^2\frac{\beta^2}{\alpha^2})^2(|X_1-Y_1|^2+|X_2-Y_2|^2+2|X_1-X_2||Y_1-Y_2|)$$

Moreover, using (13.2) and (13.7),

$$|X_1-Y_1|^2+|X_2-Y_2|^2 = (1-\mu)^2(|X_1-X_o|^2+|X_2-X_o|^2)$$

$$\leqq 2(1-\mu)^2\beta^2 \leqq 18(1-\mu)^2|X_2-X_1|^2$$

$$= \frac{1}{5}|X_2-X_1||Y_2-Y_1|$$

Formula (13.9) yields then

$$\frac{|f(X_2)-f(X_1)|}{|X_2-X_1|} \geqq \frac{m^2}{M}(1-\sigma^2\varepsilon^2\beta^2\alpha^{-2})^2 - \frac{1}{10M}(M^2-m^2(1-\sigma^2\varepsilon^2\beta^2\alpha^{-2})^2$$

$$= \frac{11}{10}m(1+\varepsilon)^{-2}(1-\sigma^2\varepsilon^2\beta^2\alpha^{-2})^2 - \frac{1}{10}m(1+\varepsilon)^2$$

Using assumption (13.13) we have then

$$\frac{|f(X_2)-f(X_1)|}{|X_2-X_1|} \geqq \frac{11}{10}m(1-\varepsilon)^2(1-2\sigma^2\varepsilon^2\beta^2\alpha^{-2}) - \frac{1}{10}m(1+\varepsilon)^2$$

$$\geqq \frac{11}{10}m(1-2\varepsilon)(1-2\sigma^2\varepsilon^2\beta^2\alpha^{-2}) + \frac{1}{2}\varepsilon^2 - \frac{1}{10}m(1+\varepsilon)^2$$

$$\geqq m(1-\frac{12}{5}\varepsilon - \frac{11}{5}\sigma^2\varepsilon^2\beta^2\alpha^{-2})$$

$$\geqq m(1-3\varepsilon-4\sigma^2\varepsilon^2\beta^2\alpha^{-2})$$

This inequality, on the other hand, is trivially satisfied when

$$\sigma \varepsilon \frac{\beta}{\alpha} > \frac{1}{2} .$$

In this way we have established (13.1) generally, taking for C the constant[1]

$$C = 36\sigma^2 = 36 \cdot 2^{10} \cdot (19\pi)^2 \tag{13.14}$$

Corollary IX.

Let R be a convex open set containing a ball of radius α and being contained in a concentric ball of radius β. There exists a universal constant γ such that any (m,M)-isometric mapping of R with

$$\varepsilon \frac{\beta}{\alpha} = \left(\sqrt{\frac{M}{m}} - 1\right)\frac{\beta}{\alpha} < \gamma \tag{13.15}$$

is uni-valent in R (that is constitutes a one-one mapping of R).

Proof:

The special case $\beta/\alpha = 1$, corresponding to a ball R, has been settled already in Corollary VII, p. 32, by showing that the mapping is uni-valent for

$$\varepsilon < 2^{1/4} - 1 = .18\cdots .$$

The general case follows from Theorem X, p. 57, but with a considerable poorer constant. We only have to take

$$\gamma = \mathrm{Min}\left(\frac{1}{4}, \frac{1}{2} C^{-1/2}\right) .$$

1. The coefficient 3 for the term with ε is chosen only for simplicity. Any number > 2 could be arrived at, taking μ sufficiently close to 1 and C sufficiently large.

For the C given by (13.14) this results in the choice of

$$\gamma = \frac{1}{3648\pi} \quad . \tag{13.16}$$

There is a <u>largest</u> constant γ such that $\varepsilon\beta/\alpha < \gamma$ implies uni-valence of the mapping. We can see that this best constant cannot exceed the value 10π. For take in the complex Z-plane for R the convex hull of the circle $|Z| < \alpha$ and of the two points $Z = \pm i\beta$ where $\alpha < \beta$. Take for f the conformal mapping provided by the analytic function

$$e^{Z\alpha^{-1}\log(1+\varepsilon)}$$

which is (m,M)-isometric in R with $m = (1+\varepsilon)^{-1}$, $M = 1+\varepsilon$. The mapping assigns the same image to the two points $Z = \pm\pi i\alpha/\log(1+\varepsilon)$. Thus the mapping is not uni-valent in R when

$$\beta > \frac{\pi\alpha}{\log(1+\varepsilon)} \quad .$$

This is certainly the case when $\varepsilon\beta/\alpha > 10\pi$, for then, since also $\beta/\alpha > 1$,

$$\begin{aligned} \frac{\beta}{\alpha}\log(1+\varepsilon) &\geqq \operatorname{Max}\left(10\pi\,\frac{\log(1+\varepsilon)}{\varepsilon},\ \log(1+\varepsilon)\right) \\ &\geqq \log(1+10\pi) > \pi \ . \end{aligned}$$

Corollary X.

Let R be an open convex set containing a ball of radius α and contained in a concentric ball of radius β. Then we have for the stiffness of R the estimate

$$s(\varepsilon,R) \geqq \frac{1}{2} - c\,\frac{\beta^2}{\alpha^2}\,\varepsilon \tag{13.17}$$

where c is a universal constant.

(In analogy to formula (12.10) one may conjecture that the constant $\frac{1}{2}$ in (13.17) can really be replaced by 1.)

Proof:

It is sufficient to prove (13.17) for the case where

$$\frac{\beta^2}{\alpha^2}\varepsilon < \lambda \tag{13.18}$$

with any fixed positive λ; the inequality (13.19) follows then without the restriction (13.18) if we replace c by Max$(c, c/\lambda)$.
By (13.1)

$$(1+\varepsilon')^2 = \frac{M'}{m'} \leqq \frac{(1+\varepsilon)^2}{1-3\varepsilon-C\beta^2\alpha^{-2}\varepsilon^2} \tag{13.19}$$

provided the denominator on the right-hand side is positive. This is certainly the case if (13.18) holds with a sufficiently small λ, for then also $\varepsilon < \lambda$ and

$$3\varepsilon + C\beta^2\alpha^{-2}\varepsilon^2 < 3\lambda + C\lambda^2 \ .$$

More precisely we find from (13.19) for λ sufficiently small an estimate of the form

$$1+2\varepsilon' \leqq (1+\varepsilon')^2 \leqq 1+4\varepsilon + O(\beta^2\alpha^{-2}\varepsilon^2)$$

This implies immediately the desired inequality

$$s(\varepsilon, R) = \frac{\varepsilon}{\varepsilon'} \geqq \frac{1}{2} - O(\beta^2\alpha^{-2}\varepsilon) \ .$$

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